37 Tutorial 6 Limits

Learn to prove the following theorems, accurately, efficiently, using quality proof machine, without having to refer to notes. Learn to prove the following theorems, accurately, efficiently, using quality proof machine, without having to refer to notes. The first step of this process is to work through each and put the reason why each line appears where it appears. The possible reasons are:

(a) (Proof type II) Assume the ifs
(b) (Proof type II) To show the thens
(c) (Rewriting) This is the definition of _______.
(d) (Proof type III) To show something exists, construct it.
(e) (Proof type III) To show the construction is valid.
(f) (Proof type I) Compute the left hand side.
(g) (Proof type I) Compute the right hand side.

Practice each proof so that you can do it efficiently without referring to notes.

37.0.1 Alternative characterization of the metric space topology

Proposition 37.1. *Let (X, d) be a strict metric space. Let*

$$\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\} \quad \textit{and let} \quad \mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \mathbb{E} \textit{ and } x \in X\},$$

the set of open balls in X. Let $\mathcal{T}$ be the metric space topology on X. Let $U \subseteq X$. Then $U \in \mathcal{T}$ if and only if

$$\textit{there exists } \mathcal{S} \subseteq \mathcal{B} \textit{ such that} \quad U = \bigcup_{B \in \mathcal{S}} B.$$

37.0.2 Interiors and closures

Proposition 37.2. *Let X be a topological space. Let $A \subseteq X$.*

(a) The interior of A is the set of interior points of A.

(b) The closure of A is the set of close points of A.

37.0.3 Limits and continuity

Theorem 37.3. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f: X \to Y$ be a function.*

(a) The function f is continuous if and only if f satisfies:

$$\textit{if } a \in X \qquad \textit{then} \quad f \textit{ is continuous at } a.$$

(b) Let $a \in X$. Then

$$f \textit{ is continuous at } a \qquad \textit{if and only if} \qquad \lim_{x \to a} f(x) = f(a).$$

(c) Let $a \in X$ such that $a \in \overline{X - \{a\}}$. Then

$$f \textit{ is continuous at } a \qquad \textit{if and only if} \qquad \lim_{\substack{x \to a \\ x \neq a}} f(x) = f(a).$$

(d) Let (X, d) be a strict metric space and let $\mathcal{T}_X$ be the metric space topology on X. Then f is continuous if and only if f satisfies:

if $(x_1, x_2, \ldots)$ is a sequence in X and

$$\textit{if } \lim_{n\to\infty} x_n \textit{ exists} \qquad \textit{then} \qquad \lim_{n\to\infty} f(x_n) = f\Big(\lim_{n\to\infty} x_n\Big).$$

37.0.4 The topology in a metric space is determined by limits of sequences

Theorem 37.4. *Let (X, d) be a strict metric space and let $A \subseteq X$ and let $\overline{A}$ be the closure of A. Then*

$$\overline{A} = \big\{z \in X \mid \textit{there exists a sequence } (a_1, a_2, \ldots) \textit{ in } A \textit{ with } z = \lim_{n\to\infty} a_n\big\}.$$

37.0.5 Limits in metric spaces

Proposition 37.5. *Let (X, d_X) and (Y, d_Y) be strict metric spaces, let $\mathcal{T}_X$ be the metric space topology on X and let $\mathcal{T}_Y$ be the metric space topology on Y. Let $f\colon X \to Y$ be a function and let $y \in Y$.*

(a) Let $a \in X$. Then $\lim_{x\to a} f(x) = y$ if and only if f satisfies

$$\begin{aligned}&\textit{if } \epsilon \in \mathbb{E} \textit{ then there exists } \delta \in \mathbb{E} \textit{ such that}\\ &\qquad \textit{if } x \in X \textit{ and } d_X(x, a) < \delta \quad \textit{then} \quad d_Y(f(x), y) < \epsilon.\end{aligned}$$

(b) Let $a \in X$ such that $a \in \overline{X - \{a\}}$. Then $\lim_{\substack{x\to a\\ x\neq a}} f(x) = y$ if and only if f satisfies

$$\begin{aligned}&\textit{if } \epsilon \in \mathbb{E} \textit{ then there exists } \delta \in \mathbb{E} \textit{ such that}\\ &\qquad \textit{if } x \in X \textit{ and } 0 < d_X(x, a) < \delta \quad \textit{then} \quad d_Y(f(x), y) < \epsilon.\end{aligned}$$

(c) Let $(x_1, x_2, \ldots)$ be a sequence in X and let $z \in X$. Then $\lim_{n\to\infty} x_n = z$ if and only if $(x_1, x_2, \ldots)$ satisfies

$$\textit{if } \varepsilon \in \mathbb{E} \textit{ then there exists } \ell \in \mathbb{Z}_{>0} \textit{ such that} \quad \textit{if } n \in \mathbb{Z}_{\geq \ell} \textit{ then } d(x_n, z) < \varepsilon.$$

38 Tutorial 6: Solutions

38.1 Alternative characterization of the metric space topology

Proposition 38.1. *Let (X,d) be a strict metric space. Let*

$$\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\} \quad \text{and let} \quad \mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \mathbb{E} \text{ and } x \in X\},$$

the set of open balls in X. Let $\mathcal{T}$ be the metric space topology on X. Let $U \subseteq X$. Then $U \in \mathcal{T}$ if and only if

$$\text{there exists } \mathcal{S} \subseteq \mathcal{B} \text{ such that} \quad U = \bigcup_{B \in \mathcal{S}} B.$$

Proof.
$\Leftarrow$: Assume $U = \bigcup_{B\in\mathcal{S}} B$.
To show: $U \in \mathcal{T}$.
To show: If $x \in U$ then there exists $\epsilon \in \mathbb{E}$ such that $B_\epsilon(x) \subseteq U$.
Assume $x \in U$.
Since $U = \bigcup_{B\in\mathcal{S}} B$ then there exists $B \in \mathcal{S}$ such that $x \in B$.
By definition of $\mathcal{B}$ there exists $\delta \in \mathbb{E}$ and $y \in X$ such that $B = B_\delta(y)$.
Since $x \in B = B_\delta(y)$ then $d(x,y) < \delta$.
Let $\epsilon = 10^{-k}$, where $k \in \mathbb{Z}_{>0}$ is such that $0 < 10^{-k} < \delta - d(x,y)$.
To show: $B_\epsilon(x) \subseteq B_\delta(y)$.
To show: If $p \in B_\epsilon(x)$ then $p \in B_\delta(y)$.
Assume $p \in B_\epsilon(x)$.
Since $d(p,y) \le d(p,x) + d(x,y) < \epsilon + d(x,y) < \delta$ then $p \in B_\delta(y)$.
So $B_\epsilon(x) \subseteq B_\delta(y) \subseteq U$.
Since $B_\delta(y) = B$ and $B \in \mathcal{S}$ then $B_\epsilon(x) \subseteq U$.
So $U \in \mathcal{T}$.

$\Rightarrow$: Assume $U \in \mathcal{T}$.
If $x \in U$ then there exists $\epsilon_x \in \mathbb{E}$ such that $B_{\epsilon_x}(x) \subseteq U$.
To show: There exists $\mathcal{S} \subseteq \mathcal{B}$ such that $U = \bigcup_{B\in\mathcal{S}} B$.
Let $\mathcal{S} = \{B_{\epsilon_x}(x) \mid x \in U\}$.
To show: $U = \bigcup_{B\in\mathcal{S}} B$.
To show: (a) $U \supseteq \bigcup_{B\in\mathcal{S}} B$.
(b) $U \subseteq \bigcup_{B\in\mathcal{S}} B$.

(a) If $B \in \mathcal{S}$ then $B = B_{\epsilon_x}(x) \subseteq U$.
So $U \supseteq \bigcup_{B\in\mathcal{S}} B$.

(b) To show: If $x \in U$ then $x \in \left(\bigcup_{B\in\mathcal{S}} B\right)$.
Assume $x \in U$.
Since $x \in B_{\epsilon_x}(x)$ and $B_{\epsilon_x}(x) \in \mathcal{S}$ then $x \in \bigcup_{B\in\mathcal{S}} B$.
So $U \subseteq \left(\bigcup_{B\in\mathcal{S}} B\right)$.

So $U = \bigcup_{B\in\mathcal{S}} B$. □

38.2 Interiors and closures

Proposition 38.2. *Let X be a topological space. Let $A \subseteq X$.*

(a) The interior of A is the set of interior points of A.

(b) The closure of A is the set of close points of A.

Proof.

(a) Let $I = \{x \in A \mid x \text{ is an interior point of } A\}$.
To show: $A^\circ = I$.
To show: (aa) $I \subseteq A^\circ$.
(ab) $A^\circ \subseteq I$.

(aa) Let $x \in I$.
Then there exists a neighborhood N of x with $N \subseteq A$.
So there exists an open set U with $x \in U \subseteq N \subseteq A$.
Since $U \subseteq A$ and U is open $U \subseteq A^\circ$.
So $x \in A^\circ$.
So $I \subseteq A^\circ$.

(ab) Assume $x \in A^\circ$.
Then A° is open and $x \in A^\circ \subseteq A$.
So x is a interior point of A.
So $x \in I$.
So $A^\circ \subseteq I$.

So $I = A^\circ$.

(b) Let $C = \{x \in X \mid \text{if } N \in \mathcal{N}(x) \text{ then } N \cap A \neq \emptyset\}$ be the set of close points of A.
Then

$$\begin{aligned} C^c &= \{x \in X \mid \text{there exists } N \in \mathcal{N}(x) \text{ such that } N \cap A = \emptyset\} \\ &= \{x \in X \mid \text{there exists } N \in \mathcal{N}(x) \text{ such that } N \subseteq A^c\}. \end{aligned}$$

which is the set of interior points of A^c.
Thus, by part (a), $C^c = (A^c)^\circ$.
So $C = ((A^c)^\circ)^c$.
To show: $C = \overline{A}$.
To show: $((A^c)^\circ)^c = \overline{A}$.

Claim: If $F \subseteq X$ then $(F^\circ)^c = \overline{F^c}$.
Let $F \subseteq X$.
Then F° is open and $(F^\circ)^c$ is closed.
Since $F^\circ \subseteq F$, then $(F^\circ)^c \supseteq F^c$.
So $(F^\circ)^c \supseteq \overline{F^c}$.
If V is closed and $V \supseteq F^c$ then V^c is open and $V^c \subseteq F$.
Thus, if V is closed and $V \supseteq F^c$ then $V^c \subseteq F^\circ$.
Thus, if V is closed and $V \supseteq F^c$ then $V \supseteq (F^\circ)^c$.
So $(F^\circ)^c = \overline{F^c}$.

Thus $((A^c)^\circ)^c = \overline{(A^c)^c}$.
Thus $C = ((A^c)^\circ)^c = \overline{(A^c)^c} = \overline{A}$.

$\square$

38.3 Limits and continuity

Theorem 38.3. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.*
Let $f: X \to Y$ be a function.

(a) f is continuous if and only if f satisfies:

$$\text{if } a \in X \qquad \text{then} \qquad f \text{ is continuous at } a.$$

(b) Let $a \in X$. Then

$$f \text{ is continuous at } a \qquad \text{if and only if} \qquad \lim_{x \to a} f(x) = f(a).$$

(c) Let $a \in X$ such that $a \in \overline{X - \{a\}}$. Then

$$f \text{ is continuous at } a \qquad \text{if and only if} \qquad \lim_{\substack{x \to a \\ x \neq a}} f(x) = f(a).$$

(d) Let (X, d) be a strict metric space and let $\mathcal{T}_X$ be the metric space topology on X. Then f is continuous if and only if f satisfies:

if $(x_1, x_2, \ldots)$ is a sequence in X and

$$\text{if } \lim_{n \to \infty} x_n \text{ exists} \qquad \text{then} \qquad \lim_{n \to \infty} f(x_n) = f\Big(\lim_{n \to \infty} x_n\Big).$$

Proof.

(a) $\Rightarrow$: To show: If f is continuous then f satisfies: if $a \in X$ then f is continuous at a.
Assume f is continuous.
To show: If $a \in X$ then f is continuous at a.
Assume $a \in X$.
To show: If $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.
Assume $N \in \mathcal{N}(f(a))$.
Then there exists $V \in \mathcal{T}_Y$ such that $f(a) \in V \subseteq N$.
To show: $f^{-1}(N) \in \mathcal{N}(a)$.
To show: There exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq f^{-1}(N)$.
Let $U = f^{-1}(V)$.
Since f is continuous then U is open in X.
Since $f(a) \in V \subseteq N$ then $a \in f^{-1}(V) = U \subseteq f^{-1}(N)$.
So $f^{-1}(N) \in \mathcal{N}(a)$.
So f is continuous at a.

(a) $\Leftarrow$: Assume that if $a \in X$ then f is continuous at a.
To show: f is continuous.
To show: If $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{T}_X$.
Assume $V \in \mathcal{T}_Y$.
To show: $f^{-1}(V)$ is open in X.
To show: If $a \in f^{-1}(V)$ then a is an interior point of $f^{-1}(V)$.
Assume $a \in f^{-1}(V)$.
To show: There exists $U \in \mathcal{N}(a)$ such that $a \in U \subseteq f^{-1}(V)$.
Since $V \in \mathcal{T}_Y$ and $f(a) \in V$ then $V \in \mathcal{N}(f(a))$.
Since f is continuous at a then $f^{-1}(V) \in \mathcal{N}(a)$.
Let $U = f^{-1}(V)$.

Then $a \in U \subseteq f^{-1}(V)$.
So a is an interior point of $f^{-1}(V)$.
So $f^{-1}(V)$ is open in X.
So f is continuous.

(b) $\Rightarrow$: To show: If f is continuous at a then $\lim_{x\to a} f(x) = f(a)$.
Assume f is continuous at a.
To show: $\lim_{x\to a} f(x) = f(a)$.
To show: If $N \in \mathcal{N}(f(a))$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.
Assume $N \in \mathcal{N}(f(a))$.
To show: There exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.
Since f is continuous at a and $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.
Let $P = f^{-1}(N)$.
Then $f(P) = f(f^{-1}(N)) \subseteq N$.
So $\lim_{x\to a} f(x) = f(a)$.

(b) $\Leftarrow$: To show: If $\lim_{x\to a} f(x) = f(a)$ then f is continuous at a.
Assume $\lim_{x\to a} f(x) = f(a)$.
To show: f is continuous at a.
To show: If $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.
Assume $N \in \mathcal{N}(f(a))$.
To show: $f^{-1}(N) \in \mathcal{N}(a)$.
To show: There exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq f^{-1}(N)$.
Since $\lim_{x\to a} f(x) = f(a)$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.
So $f^{-1}(N) \supseteq P$.
Since $P \in \mathcal{N}(a)$, there exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq P$.
So there exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq P \subseteq f^{-1}(N)$.
So $f^{-1}(N) \in \mathcal{N}(a)$.
So f is continuous at a.

(c) $\Rightarrow$: Assume $a \in \overline{X - \{a\}}$.
To show: If f is continuous at a then $\lim\limits_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$.
Assume f is continuous at a.
To show: $\lim\limits_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$.
To show: If $N \in \mathcal{N}(f(a))$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P - \{a\})$.
Assume $N \in \mathcal{N}(f(a))$.
To show: There exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P - \{a\})$.
Since f is continuous at a and $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.
Let $P = f^{-1}(N)$.
Then $f(P - \{a\}) \subseteq f(P) = f(f^{-1}(N)) \subseteq N$.
So $\lim\limits_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$.

(c) $\Leftarrow$: Assume $a \in \overline{X - \{a\}}$.
To show: If $\lim\limits_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$ then f is continuous at a.
Assume $\lim\limits_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$.
To show: f is continuous at a.
To show: If $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.
Assume $N \in \mathcal{N}(f(a))$.
To show: $f^{-1}(N) \in \mathcal{N}(a)$.
To show: There exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq f^{-1}(N)$.

Since $\lim\limits_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$ there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P - \{a\})$.
So $f^{-1}(N) \supseteq P - \{a\}$.
Since $N \in \mathcal{N}(f(a))$ then $f(a) \in N$ and $a \in f^{-1}(N)$.
So $f^{-1}(N) \supseteq P$.
Since $P \in \mathcal{N}(a)$, there exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq P$.
So there exists $U \in \mathcal{T}_X$ such that $a \in U \subseteq P \subseteq f^{-1}(N)$.
So $f^{-1}(N) \in \mathcal{N}(a)$.
So f is continuous at a.

(d) $\Rightarrow$: Assume f is continuous.
To show: f satisfies

$$\text{if } (x_1, x_2, \ldots) \text{ is a sequence in } X \text{ and } \lim_{n\to\infty} x_n \text{ exists then } f\Big(\lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} f(x_n). \qquad (*)$$

Assume $(x_1, x_2, \ldots)$ is a sequence in X and $\lim_{n\to\infty} x_n = a$.
To show: $f(a) = \lim\limits_{n\to\infty} f(x_n)$.
To show: If $N \in \mathcal{N}(f(a))$ then there exists $t \in \mathbb{Z}_{>0}$ such that $N \supseteq (f(x_t), f(x_{t+1}), \ldots)$.
Assume $N \in \mathcal{N}(f(a))$.
Since f is continuous then $f^{-1}(N) \in \mathcal{N}(a)$.
Since $\lim_{n\to\infty} x_n = a$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $f^{-1}(N) \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.
Let $t = \ell$.
Then $f^{-1}(N) \supseteq \{x_t, x_{t+1}, \ldots\}$.
So $N \supseteq \{f(x_t), f(x_{t+1}), \ldots\}$.
So f satisfies (*).

(d) $\Leftarrow$: To show: If f is not continuous then f does not satisfy (*).
Assume f is not continuous.
Then there exists a such that f is not continuous at a.
So there exists $N \in \mathcal{N}(f(a))$ such that $f^{-1}(N) \notin \mathcal{N}(a)$.
To show: There exists a sequence $(x_1, x_2, \ldots)$ such that $\lim_{n\to\infty} x_n$ exists and $\lim_{n\to\infty} f(x_n) \neq f\big(\lim_{n\to\infty} x_n\big)$.
Since $f^{-1}(N) \notin \mathcal{N}(a)$ then $f^{-1}(N) \not\supseteq B_{10^{-\ell}}(a)$, for $\ell \in \mathbb{Z}_{>0}$. Let

$$x_1 \in B_{10^{-1}}(a) \cap f^{-1}(N)^c, \quad x_2 \in B_{10^{-2}}(a) \cap f^{-1}(N)^c, \quad \ldots.$$

To show: (da) $\lim_{n\to\infty} x_n = a$.
(db) $\lim_{n\to\infty} f(x_n) \neq f(a)$.

(da) To show: If $P \in \mathcal{N}(a)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $x_n \in P$.
Assume $P \in \mathcal{N}(a)$.
To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that $P \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.
Since $P \in \mathcal{N}(a)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $P \supseteq B_{10^{-\ell}}(a)$.
To show: $P \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.
To show: If $n \in \mathbb{Z}_{\geq \ell}$ then $x_n \in P$.
Assume $n \in \mathbb{Z}_{\geq \ell}$.
Since $n \geq \ell$ then $10^{-\ell} \leq 10^{-n}$ and $x_n \in B_{10^{-n}}(a) \subseteq B_{10^{-\ell}}(a) \subseteq P$.
So $P \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.
So $\lim_{n\to\infty} x_n = a$.

(db) To show: $\lim_{n\to\infty} f(x_n) \neq f(a)$.
To show: There exists $M \in \mathcal{N}(f(a))$ such that $\{j \in \mathbb{Z}_{>0} \mid f(x_j) \in M^c\}$ is infinite.
Let $M = N$.
To show: $\{j \in \mathbb{Z}_{>0} \mid f(x_j) \in N^c\}$ is infinite.
Since $x_j \in f^{-1}(N)^c$ then $f(x_j) \notin N$, for $j \in \mathbb{Z}_{>0}$.

So $\{f(x_1), f(x_2), \ldots\} \subseteq N^c$.
So $\{j \in \mathbb{Z}_{>0} \mid f(x_j) \in N^c\}$ is infinite.
So $\lim_{n\to\infty} f(x_n) \neq f(a)$.

So f does not satisfy (*).

□

To change the proof of (d) above to a proof for first countable topological spaces $(X, \mathcal{T}_X)$, replace the use of the open balls $B_{10^{-1}}(a) \supseteq B_{10^{-2}}(a) \supseteq \cdots$ by generators $B_1 \supseteq B_2 \supseteq \cdots$ of $\mathcal{N}(a)$, the neighborhood filter of a.

38.4 The topology in a metric space is determined by limits of sequences

Theorem 38.4. *Let (X, d) be a strict metric space and let $A \subseteq X$ and let $\overline{A}$ be the closure of A. Then*

$$\overline{A} = \big\{z \in X \mid \text{there exists a sequence } (a_1, a_2, \ldots) \text{ in } A \text{ with } z = \lim_{n\to\infty} a_n\big\}.$$

Proof. Let $R = \big\{z \in X \mid \text{there exists a sequence } (a_1, a_2, \ldots) \text{ in } A \text{ with } z = \lim_{n\to\infty} a_n\big\}$.

To show: (a) $R \subseteq \overline{A}$.
(b) $\overline{A} \subseteq R$.

(a) To show: If $z \in R$ then $z \in \overline{A}$.
Assume $z \in R$.
To show: $z \in \overline{A}$.
We know there exists a sequence $(a_1, a_2, \ldots)$ in A with $z = \lim_{n\to\infty} a_n$.
To show: z is a close point of A.
To show: If N is a neighborhood of z then $N \cap A \neq \emptyset$.
Assume N is a neighborhood of z.
Since $\lim_{n\to\infty} a_n = z$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in N$.
So $N \cap A \neq \emptyset$.
So z is a close point of A.
So $R \subseteq \overline{A}$.

(b) To show: $\overline{A} \subseteq R$.
To show: If $z \in \overline{A}$ then $z \in R$.
Let $z \in \overline{A}$.
To show: $z \in R$.
To show: There exists a sequence $(a_1, a_2, \ldots)$ in A with $z = \lim_{n\to\infty} a_n$.
Using that z is a close point of A,

$$\text{let} \quad a_1 \in B_{0.1}(z) \cap A, \quad a_2 \in B_{0.01}(z) \cap A, \quad a_3 \in B_{0.001}(z) \cap A, \quad \ldots.$$

To show: $z = \lim_{n\to\infty} a_n$.
To show: If P is a neighborhood of z then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in P$.
Let P be a neighborhood of z.
Then there exists $\ell \in \mathbb{Z}_{>0}$ such that $B_{10^{-\ell}}(z) \subseteq P$.
To show: If $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in P$.
Assume $n \in \mathbb{Z}_{\geq \ell}$.
Since $n \geq \ell$ then $10^{-n} \leq 10^{-\ell}$ and

$$a_n \in B_{10^{-n}}(z) \subseteq B_{10^{-\ell}}(z) \subseteq P,$$

So $\lim_{n\to\infty} a_n = z$.
So $z \in R$.
So $\overline{A} \subseteq R$.

□

To change the proof of (b) above to a proof for first countable topological spaces $(X, \mathcal{T}_X)$, replace the use of the open balls $B_{10^{-1}}(a) \supseteq B_{10^{-2}}(a) \supseteq \cdots$ by generators $B_1 \supseteq B_2 \supseteq \cdots$ of $\mathcal{N}(a)$, the neighborhood filter of a.

38.5 Limits in metric spaces

Proposition 38.5. *Let (X, d_X) and (Y, d_Y) be strict metric spaces, let $\mathcal{T}_X$ be the metric space topology on X and let $\mathcal{T}_Y$ be the metric space topology on Y. Let $f\colon X \to Y$ be a function and let $y \in Y$.*

(a) Let $a \in X$. Then $\lim_{x\to a} f(x) = y$ if and only if f satisfies

$$\text{if } \epsilon \in \mathbb{E} \text{ then there exists } \delta \in \mathbb{E} \text{ such that} \\ \text{if } x \in X \text{ and } d_X(x,a) < \delta \quad \text{then} \quad d_Y(f(x), y) < \epsilon.$$

(b) Let $a \in X$ such that $a \in \overline{X - \{a\}}$. Then $\lim_{\substack{x\to a\\ x\neq a}} f(x) = y$ if and only if f satisfies

$$\text{if } \epsilon \in \mathbb{E} \text{ then there exists } \delta \in \mathbb{E} \text{ such that} \\ \text{if } x \in X \text{ and } 0 < d_X(x,a) < \delta \quad \text{then} \quad d_Y(f(x), y) < \epsilon.$$

(c) Let $(x_1, x_2, \ldots)$ be a sequence in X and let $z \in X$. Then $\lim_{n\to\infty} x_n = z$ if and only if $(x_1, x_2, \ldots)$ satisfies

$$\text{if } \varepsilon \in \mathbb{E} \text{ then there exists } \ell \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } n \in \mathbb{Z}_{\geq \ell} \text{ then } d(x_n, z) < \varepsilon.$$

Proof. (a) By definition, $\lim_{x\to a} f(x) = y$ if and only if f satisfies: if $N \in \mathcal{N}(y)$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.

By definition of the metric space topology, $N \in \mathcal{N}(y)$ if and only if there exists $\epsilon \in \mathbb{E}$ such that $B_\epsilon(y) \subseteq N$.

Thus $\lim_{x\to a} f(x) = y$ if and only if f satisfies: if $B_\epsilon(y)$ is an open ball at y then there exists $B_\delta(a)$, an open ball at a such that $B_\epsilon(y) \supseteq f(B_\delta(a))$.

By definition, $B_\delta(a) = \{x \in X \mid d(x,a) < \delta\}$.

Thus, $\lim_{x\to a} f(x) = y$ if and only if f satisfies: if $\varepsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $d_X(x,a) < \delta$ then $d_Y(f(x), y) < \varepsilon$.

(b) By definition, $\lim_{\substack{x\to a\\ x\neq a}} f(x) = y$ if and only if f satisfies: if $N \in \mathcal{N}(y)$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P - \{a\})$.

By definition of the metric space topology, $N \in \mathcal{N}(y)$ if and only if there exists $\epsilon \in \mathbb{E}$ such that $B_\epsilon(y) \subseteq N$.

Thus $\lim_{\substack{x\to a\\ x\neq a}} f(x) = y$ if and only if f satisfies: if $B_\epsilon(y)$ is an open ball at y then there exists $B_\delta(a)$, an open ball at a such that $B_\epsilon(y) \supseteq f(B_\delta(a) - \{a\})$.

By definition, $B_\epsilon(y) = \{x \in Y \mid d(x,y) < \epsilon\}$ and $B_\delta(a) - \{a\} = \{x \in X \mid 0 < d(x,a) < \delta\}$.

Thus, $\lim_{\substack{x\to a\\ x\ne a}} f(x) = y$ if and only if f satisfies: if $\epsilon \in \mathbb{E}$ then there exists $\delta \in \mathbb{E}$ such that if $x \in X$ and $0 < d_X(x,a) < \delta$ then $d_Y(f(x), y) < \epsilon$.

(c) By definition, $\lim_{n\to\infty} x_n = z$ if and only if $(x_1, x_2, \ldots)$ satisfies: if $P \in \mathcal{N}(z)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $P \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.

By definition of the metric space topology, $P \in \mathcal{N}(y)$ if and only if there exists $\epsilon \in \mathbb{E}$ such that $B_\epsilon(y) \subseteq P$.

So $\lim_{n\to\infty} x_n = z$ if and only if $(x_1, x_2, \ldots)$ satisfies: if $\epsilon \in \mathbb{E}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $B_\epsilon(z) \supseteq \{x_\ell, x_{\ell+1}, \ldots, \}$.

By definition, $B_\epsilon(a) = \{x \in X \mid d(x,a) < \epsilon\}$.

Thus, $\lim_{n\to\infty} x_n = z$ if and only if $(x_1, x_2, \ldots)$ satisfies: if $\epsilon \in \mathbb{E}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\ge \ell}$ then $d(x_n, z) < \epsilon$. $\square$