29 Tutorial 6: Completions

Learn to prove the following theorems, accurately, efficiently, using quality proof machine, without having to refer to notes. Learn to prove the following theorems, accurately, efficiently, using quality proof machine, without having to refer to notes. The first step of this process is to work through each and put the reason why each line appears where it appears. The possible reasons are:

(a) (Proof type II) Assume the ifs
(b) (Proof type II) To show the thens
(c) (Rewriting) This is the definition of _______.
(d) (Proof type III) To show something exists, construct it.
(e) (Proof type III) To show the construction is valid.
(f) (Proof type I) Compute the left hand side.
(g) (Proof type I) Compute the right hand side.

Practice each proof so that you can do it efficiently without referring to notes.

29.0.1 Absolute convergence and completeness

Theorem 29.1. *Let $(V, \|\ \|)$ be a normed vector space and let $d\colon V \times V \to \mathbb{R}_{\geq 0}$ be the metric on V given by $d(x,y) = \|x-y\|$. Then V is a complete metric space if and only if V satisfies*

$$\begin{array}{c}\text{If } (a_1, a_2, \ldots) \text{ is a sequence in } V \text{ and } \displaystyle\sum_{i\in\mathbb{Z}_{>0}} \|a_i\| \text{ converges} \\ \text{then } \displaystyle\sum_{i\in\mathbb{Z}_{>0}} a_i \text{ converges.}\end{array} \qquad (*)$$

29.0.2 Construction of the completion of a metric space

Theorem 29.2. *Let (X,d) be a metric space. Let $(\widehat{X}, \hat{d}, \varphi)$ be the metric space*

$$\widehat{X} = \{\text{Cauchy sequences } \vec{x} \text{ in } X\} \qquad \text{with the function} \qquad \begin{array}{rccc}\varphi\colon & X & \longrightarrow & \widehat{X} \\ & x & \longmapsto & (x,x,x,\ldots)\end{array}$$

where $\widehat{X}$ has the metric

$$\hat{d}\colon \widehat{X} \times \widehat{X} \to \mathbb{R}_{\geq 0} \qquad \text{defined by} \qquad \hat{d}(\vec{x},\vec{y}) = \lim_{n\to\infty} d(x_n, y_n),$$

and Cauchy sequences $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$ are equal in $\widehat{X}$,

$$\vec{x} = \vec{y} \qquad \text{if } \lim_{n\to\infty} d(x_n, y_n) = 0.$$

Then $(\widehat{X}, \hat{d})$ with the isometry $\iota\colon X \to \widehat{X}$ such that

$$(\widehat{X}, \hat{d}) \text{ is a complete metric space} \qquad \text{and} \qquad \overline{\varphi(X)} = \widehat{X},$$

where $\overline{\varphi(X)}$ is the closure of the image of φ.

29.0.3 If W is complete then $B(V,W)$ is complete

Theorem 29.3. *Let $(V, \| \ \|)$ and $(W, \| \ \|)$ be normed vector spaces and let*

$$B(V,W) = \{\text{linear transformations } T\colon V \to W \mid \|T\| < \infty\} \qquad \text{where}$$

$$\|T\| = \sup\left\{ \frac{\|Tv\|}{\|v\|} \mid v \in V \text{ and } v \neq 0 \right\}.$$

If W is complete then $B(V,W)$ is complete.

30 Tutorial 6: Solutions

30.1 Absolute convergence and completeness

Theorem 30.1. *Let $(V, \| \ \|)$ be a normed vector space and let $d\colon V \times V \to \mathbb{R}_{\geq 0}$ be the metric on V given by $d(x,y) = \|x-y\|$. Then V is a complete metric space if and only if V satisfies*

$$\text{If } (a_1, a_2, \ldots) \text{ is a sequence in } V \text{ and } \sum_{i\in\mathbb{Z}_{>0}} \|a_i\| \text{ converges then } \sum_{i\in\mathbb{Z}_{>0}} a_i \text{ converges.} \qquad (*)$$

Proof. $\Rightarrow$: Assume V is complete.

To show: If $(a_1, a_2, \ldots)$ is a sequence in V and $\sum_{i\in\mathbb{Z}_{>0}} \|a_i\|$ converges then $\sum_{i\in\mathbb{Z}_{>0}} a_i$ converges.

Assume $(a_1, a_2, \ldots)$ is a sequence in V and $\sum_{i\in\mathbb{Z}_{>0}} \|a_i\|$ converges.

To show: $\sum_{i\in\mathbb{Z}_{>0}} a_i$ converges.

Let

$$s_n = \sum_{i=1}^n a_i \qquad \text{and} \qquad S_n = \sum_{i=1}^n \|a_n\|.$$

Since the sequence $(S_1, S_2, \ldots)$ converges, the sequence $(S_1, S_2, \ldots)$ is Cauchy.

Since

$$\|s_n - s_m\| = \Big\| \sum_{i=m+1}^n a_i \Big\| \leq \sum_{i=m+1}^n \|a_i\| = \|S_n - S_m\|,$$

the sequence $(s_1, s_2, \ldots)$ is Cauchy.

Since V is complete, the sequence $(s_1, s_2, \ldots)$ converges.

So $\sum_{i\in\mathbb{Z}_{>0}} a_i$ converges.

$\Leftarrow$: Assume that V satisfies (*).

To show: V is complete.

Let $(s_1, s_2, \ldots)$ be a Cauchy sequence in V.

To show: $(s_1, s_2, \ldots)$ converges. Let $k_n \in \mathbb{Z}_{>0}$ be such that if $r, m \in \mathbb{Z}_{\geq k_n}$ then $\|s_r - s_m\| < \frac{1}{2^n}$.

Let

$$a_1 = s_{k_1}, \quad a_2 = s_{k_2} - s_{k_1}, \quad a_3 = s_{k_3} - s_{k_2}, \quad \ldots. \qquad \text{Then } \|a_n\| < \tfrac{1}{2^n}.$$

So

$$\sum_{n\in\mathbb{Z}_{>0}} \|a_n\| < \sum_{n\in\mathbb{Z}_{>0}} \frac{1}{2^n} = 1.$$

So $\sum_{n\in\mathbb{Z}_{>0}} \|a_n\|$ converges.

Since V satisfies (*) then $\sum_{n\in\mathbb{Z}_{>0}} a_n$ converges.

So the sequence $(s_{k_1}, s_{k_2}, \ldots)$ converges since

$$s_{k_1} = a_1, \quad s_{k_2} = a_1 + a_2, \quad s_{k_3} = a_1 + a_2 + a_3, \quad \ldots.$$

So the sequence $(s_1, s_2, \ldots)$ has a cluster point.
Since $(s_1, s_2, \ldots)$ is Cauchy and has a cluster point then $(s_1, s_2, \ldots)$ converges.
So V is complete.

□

30.2 Construction of the completion of a metric space

Theorem 30.2. *Let (X, d) be a metric space. Let $(\widehat{X}, \hat{d}, \varphi)$ be the metric space*

$$\widehat{X} = \{\textit{Cauchy sequences } \vec{x} \textit{ in } X\} \qquad \textit{with the function} \quad \begin{array}{cccc} \varphi\colon & X & \longrightarrow & \widehat{X} \\ & x & \longmapsto & (x, x, x, \ldots) \end{array}$$

where $\widehat{X}$ has the metric

$$\hat{d}\colon \widehat{X} \times \widehat{X} \to \mathbb{R}_{\geq 0} \qquad \textit{defined by} \qquad \hat{d}(\vec{x}, \vec{y}) = \lim_{n\to\infty} d(x_n, y_n),$$

and Cauchy sequences $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$ are equal in $\widehat{X}$,

$$\vec{x} = \vec{y} \qquad \textit{if } \lim_{n\to\infty} d(x_n, y_n) = 0.$$

Then $(\widehat{X}, \hat{d})$ with the isometry $\iota\colon X \to \widehat{X}$ such that

$$(\widehat{X}, \hat{d}) \textit{ is a complete metric space} \qquad \textit{and} \qquad \overline{\varphi(X)} = \widehat{X},$$

where $\overline{\varphi(X)}$ is the closure of the image of φ.

Proof.
To show: (a) $(\widehat{X}, \hat{d})$ is a metric space.
(b) $(\widehat{X}, \hat{d})$ is complete.
(c) $\varphi\colon X \to \widehat{X}$ is an isometry.
(d) $\overline{\varphi(X)} = \widehat{X}$.

(c) To show: If $x, y \in X$ then $\hat{d}(\varphi(x), \varphi(y)) = d(x, y)$.
Assume $x, y \in X$.

$$\hat{d}(\varphi(x), \varphi(y)) = \lim_{nto\infty} d(\varphi(x)_n, \varphi(y)_n) = \lim_{n\to\infty} d(x, y) = d(x, y).$$

So φ is an isometry.

(a) To show: $(\widehat{X}, \hat{d})$ is a metric space.
To show: (aa) $\hat{d}\colon \widehat{X} \times \widehat{X} \to \mathbb{R}_{\geq 0}$ given by $\hat{d}(\vec{x}, \vec{y}) = \lim_{n\to\infty} d(x_n, y_n)$ is a function.
(ab) If $\vec{x}, \vec{y} \in \widehat{X}$ then $\hat{d}(\vec{x}, \vec{y}) = \hat{d}(\vec{y}, \vec{x})$.
(ac) If $\vec{x} \in \widehat{X}$ then $\hat{d}(\vec{x}, \vec{x}) = 0$.
(ad) If $\vec{x}, \vec{y} \in \widehat{X}$ and $\hat{d}(\vec{x}, \vec{y}) = 0$ then $\vec{x} = \vec{y}$.
(ab) If $\vec{x}, \vec{y}, \vec{z} \in \widehat{X}$ then $\hat{d}(\vec{x}, \vec{y}) \leq \hat{d}(\vec{x}, \vec{y}) + \hat{d}(\vec{y}, \vec{z})$.

(aa) To show: If $\vec{x}, \vec{y} \in \widehat{X}$ then there exists a unique $z \in \mathbb{R}_{\geq 0}$ such that $z = \lim_{n\to\infty} d(x_n, y_n)$.
Assume $\vec{x}, \vec{y} \in \widehat{X}$ with $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$.
Let $d_1, d_2, \ldots$ be the sequence in $\mathbb{R}_{\geq 0}$ given by

$$d_n = d(x_n, y_n).$$

To show: There exists $z \in \mathbb{R}_{\geq 0}$ such that $z = \lim_{n\to\infty} d_n$.
Since $\mathbb{R}_{\geq 0}$ is a metric space, and metric spaces are Hausdorff, HERE WE USE THAT METRIC SPACES ARE HAUSDORFFand limits in Hausdorff spaces are unique when they exist, the limit z will be unique if it exists.
To show: $d_1, d_2, \ldots$ is a Cauchy sequence in $\mathbb{R}_{\geq 0}$. This will show that z exists since $\mathbb{R}_{\geq 0}$ is complete HERE WE USE THAT $\mathbb{R}_{>0}$ IS A COMPLETE METRIC SPACE and Cauchy sequences in complete spaces converge.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\geq N}$ then $|d_m - d_n| < \epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
Let $N = \max(N_1, N_2)$, where

$$N_1 \text{ is such that if } n, m \in \mathbb{Z}_{\geq N_1} \text{ then } d(x_m, x_n) \in \tfrac{\epsilon}{2}, \text{ and}$$
$$N_2 \text{ is such that if } n, m \in \mathbb{Z}_{\geq N_2} \text{ then } d(y_m, y_n) \in \tfrac{\epsilon}{2}.$$

(N_1 and N$_2$ exist since $\vec{x}$ and $\vec{y}$ are Cauchy sequences.)
Assume $m, n \in \mathbb{Z}_{\geq N}$.
To show: $|d_m - d_n| < \epsilon$.

$$|d_m - d_n| = \big|d(x_m, y_m) - d(x_n, y_n)\big| \leq \big|d(x_n, x_m) + d(y_n, y_m)\big|,$$

since $d(x_n, y_n) \leq d(x_n, x_m) + d(x_n, y_n) + d(y_n, y_m)$.
So

$$|d_m - d_n| \leq \big|d(x_n, x_m) + d(y_n, y_m)\big| \leq \big|d(x_n, x_m)\big| + \big|d(y_n, y_m)\big| < \epsilon_2 + \epsilon_2 = \epsilon.$$

So $d_1, d_2, \ldots$ is a Cauchy sequence in $\mathbb{R}_{\geq 0}$.
So $z = \lim_{n\to\infty} d_n$ exists in $\mathbb{R}_{\geq 0}$.

(ab) To show: If $\vec{x}, \vec{y} \in \widehat{X}$ then $\hat{d}(\vec{x}, \vec{y}) = \hat{d}(\vec{y}, \vec{x})$.
Assume $\vec{x}, \vec{y} \in \widehat{X}$ with $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$.
Since $d(x_n, y_n) = d(y_n, x_n)$,

$$\hat{d}(\vec{x}, \vec{y}) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = \hat{d}(\vec{y}, \vec{x}).$$

(ac) To show: If $\vec{x} \in \widehat{X}$ then $\hat{d}(\vec{x}, \vec{x}) = 0$.
Assume $\vec{x} \in \widehat{X}$.
To show $\hat{d}(\vec{x}, \vec{x}) = 0$.
Since $d(x_n, x_n) = 0$,

$$\hat{d}(\vec{x}, \vec{x}) = \lim_{n\to\infty} d(x_n, x_n) = \lim_{n\to\infty} 0 = 0.$$

(ad) If $\vec{x}, \vec{y}, \vec{z} \in \widehat{X}$ then $\hat{d}(\vec{x}, \vec{y}) \leq \hat{d}(\vec{x}, \vec{z}) + \hat{d}(\vec{z}, \vec{y})$.
Assume $\vec{x}, \vec{y}, \vec{z} \in \widehat{X}$.
To show: $\hat{d}(\vec{x}, \vec{y}) \leq \hat{d}(\vec{x}, \vec{z}) + \hat{d}(\vec{z}, \vec{y})$.

$$\begin{aligned}\hat{d}(\vec{x}, \vec{y}) &= \lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty} \big(d(x_n, z_n) + d(z_n, y_n)\big) \\ &= \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n) = \hat{d}(\vec{x}, \vec{z}) + \hat{d}(\vec{z}, \vec{y}),\end{aligned}$$

where the next to last equality follows from the continuity of addition in $\mathbb{R}_{\geq 0}$.

(d) To show: $\overline{\varphi(X)} = \widehat{X}$.
To show: If $\vec{z} \in \widehat{X}$ then there exists a sequence $\vec{x}_1, \vec{x}_2, \ldots$ in $\varphi(X)$ such that $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.
Assume $\vec{z} = (z_1, z_2, \ldots) \in \widehat{X}$.
To show: There exists $\vec{x}_1, \vec{x}_2, \ldots$ in $\varphi(X)$ with $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.
Let

$$\begin{aligned} \vec{x}_1 &= (z_1, z_1, z_1, z_1, \ldots) = \varphi(z_1), \\ \vec{x}_2 &= (z_2, z_2, z_2, z_2, \ldots) = \varphi(z_1), \\ \vec{x}_3 &= (z_3, z_3, z_3, z_3, \ldots) = \varphi(z_1), \qquad \ldots \end{aligned}$$

so that $\vec{x}_1, \vec{x}_2, \ldots$ is the sequence $\varphi(z_1), \varphi(z_2), \ldots$ in $\varphi(X)$.
To show: $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.
To show: $\lim_{n\to\infty} \hat{d}(\vec{x}_n, \vec{z}) = 0$.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $n \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $\hat{d}(\vec{x}_n, \vec{z}) < \epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
Let $N \in \mathbb{Z}_{>0}$ be such that if $r, s \in \mathbb{Z}_{\geq N}$ then $d(z_r, z_s) < \epsilon/2$.
The value N exists since $\vec{z} = (z_1, z_2, \ldots)$ is a Cauchy sequence in X.
To show: If $n \in \mathbb{Z}_{\geq N}$ then $\hat{d}(\vec{x}_n, \vec{z}) < \epsilon$.
Assume $n \in \mathbb{Z}_{\geq N}$.
To show: $\hat{d}(\vec{x}_n, \vec{z}) < \epsilon$.
To show: $\lim\limits_{k\to\infty} d((\vec{x}_n)_k, z_k) < \epsilon$.

$$\lim_{k\to\infty} d((\vec{x}_n)_k, z_k) = \lim_{k\to\infty} d(z_n, z_k) \leq \frac{\epsilon}{2} < \epsilon, \qquad \text{since } d(z_n, z_k) < \tfrac{\epsilon}{2} \text{ for } k > N.$$

So $\lim\limits_{n\to\infty} \vec{x}_n = \vec{z}$.
So $\overline{\varphi(X)} = \widehat{X}$.

(b) To show: $(\widehat{X}, \hat{d})$ is complete.
To show: If $\vec{x}_1, \vec{x}_2, \ldots$ is a Cauchy sequence in $\widehat{X}$ then $\vec{x}_1, \vec{x}_2, \ldots$ converges.
Assume

$$\begin{aligned} \vec{x}_1 &= (x_{11}, x_{12}, x_{13}, \ldots), \\ \vec{x}_2 &= (x_{21}, x_{22}, x_{23}, \ldots), \\ \vec{x}_3 &= (x_{31}, x_{32}, x_{33}, \ldots), \\ &\vdots \end{aligned}$$

is a Cauchy sequence in $\widehat{X}$.
To show: There exists $\vec{z} = (z_1, z_2, \ldots)$ in $\widehat{X}$ such that $\lim\limits_{n\to\infty} \vec{x}_n = \vec{z}$.
Using that $\overline{\varphi(X)} = \widehat{X}$, for $k \in \mathbb{Z}_{>0}$ let $z_k \in X$ be such that $\hat{d}(\varphi(z_k), \vec{x}_k) < \frac{1}{k}$.

$$\begin{array}{lll} \vec{x}_1 = (x_{11}, x_{12}, x_{13}, \ldots), & \varphi(z_1) = (z_1, z_1, z_1, z_1, \ldots), & \hat{d}(\varphi(z_1), \vec{x}_1) < 1, \\ \vec{x}_2 = (x_{21}, x_{22}, x_{23}, \ldots), & \varphi(z_2) = (z_2, z_2, z_2, z_2, \ldots), & \hat{d}(\varphi(z_2), \vec{x}_2) < \frac{1}{2}, \\ \vec{x}_3 = (x_{31}, x_{32}, x_{33}, \ldots), & \varphi(z_3) = (z_3, z_3, z_3, z_3, \ldots), & \hat{d}(\varphi(z_3), \vec{x}_3) < \frac{1}{3}, \\ \quad\vdots & \quad\vdots & \quad\vdots \end{array}$$

To show: (ba) $\vec{z} = (z_1, z_2, z_3, \ldots)$ is a Cauchy sequence.
(bb) $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.

(ba) To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(z_r, z_s) < \epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(z_r, z_s) < \epsilon$.
Let $\ell_1 = \left\lceil \frac{3}{\epsilon} \right\rceil + 1$, so that $\frac{1}{\ell_1} < \frac{\epsilon}{3}$.
Let $\ell_2 \in \mathbb{Z}_{>0}$ be such that if $r, s \in \mathbb{Z}_{\geq \ell_2}$ then $\hat{d}(\vec{x}_r, \vec{x}_s) < \frac{\epsilon}{3}$.
Let $\ell = \max\{\ell_1, \ell_2\}$.
To show: If $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(z_r, z_s) < \epsilon$.
Assume $r, s \in \mathbb{Z}_{\geq \ell}$.
To show: $d(z_r, z_s) < \epsilon$.

$$\begin{aligned} d(z_r, z_s) &= \hat{d}(\varphi(z_r), \varphi(z_s)) \leq \hat{d}(\varphi(z_r), \vec{x}_r) + \hat{d}(\vec{x}_r, \vec{x}_s) + \hat{d}(\vec{x}_s, \varphi(z_s)) \\ &\leq \frac{1}{r} + \frac{\epsilon}{3} + \frac{1}{s} < \frac{1}{\ell_1} + \frac{\epsilon}{3} + \frac{1}{\ell_1} = \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

So $\vec{z}$ is a Cauchy sequence.

(bb) To show $\lim_{n\to\infty} \hat{d}(\vec{x}_n, \vec{z}) = 0$.

$$\begin{aligned} \lim_{n\to\infty} \hat{d}(\vec{x}_n, \vec{z}) &\leq \lim_{n\to\infty} \big(\hat{d}(\vec{x}_n, \varphi(z_n)) + \hat{d}(\varphi(z_n), \vec{z})\big) \leq \lim_{n\to\infty} \big(\tfrac{1}{n} + \hat{d}(\varphi(z_n), \vec{z})\big) \\ &= \lim_{n\to\infty} \tfrac{1}{n} + \lim_{n\to\infty} \hat{d}(\varphi(z_n), \vec{z}) = 0 + 0 = 0. \end{aligned}$$

So $(\widehat{X}, \hat{d})$ is complete.

So $(\widehat{X}, \hat{d})$ with $\varphi \colon X \to \widehat{X}$ is a completion of X.

□