MAST30026 Metric and Hilbert Spaces

Sample exam 1

Question 1. Consider the map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = \tfrac{1}{10}(8x+8y, x+y).$$

Recall metrics

$$\begin{aligned} d_1((x_1,y_1),(x_2,y_2) &= |x_1-x_2| + |y_1-y_2|, \\ d_2((x_1,y_1),(x_2,y_2) &= (|x_1-x_2|^2 + |y_1-y_2|^2)^{1/2}, \\ d_\infty((x_1,y_1),(x_2,y_2) &= \max\{|x_1-x_2|, |y_1-y_2|\} \end{aligned}$$

If f a contraction with respect to d_1? d_2? d_∞? Prove that your answers are correct.

Question 2. A family $\{F_i\}_{i\in I}$ is said to have the **finite intersection property** if for every finite subset J of I, $\bigcap_{i\in J} F_i = \emptyset$. Show that X is compact if and only if for every family $\{F_i\}_{i\in I}$ of closed subsets of X having the finite intersection property, the intersection $\bigcap_{i\in I} F_i \neq \emptyset$.

Question 3. Let X be a connected topological space. Let $f\colon X \to \mathbb{R}$ be continuous with $f(X) \subseteq \mathbb{Q}$. Show that f is a constant function.

Question 4. Let $[a_{ij}]$ be a infinite complex matrix, $i, j = 1, 2, \ldots$, such that if $j \in \mathbb{Z}_{>0}$ then

$$c_j = \sum_i |a_{ij}| \quad \text{converges,} \qquad \text{and} \qquad c = \sup\{c_1, c_2, \ldots\} < \infty.$$

Show that the operator $T\colon \ell^1 \to \ell^1$ defined by

$$T(b_1, b_2, \ldots) = \Big(\sum_j a_1 j b_j, \sum_j a_{2j} b_j, \ldots\Big)$$

is a bounded linear operator and that $\|T\| = c$.

Question 5. Let (X, d) be a metric space. Show that the metric $d'\colon X \times X \to \mathbb{R}$ given by

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

is equivalent to d.

Question 6. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $\{f_n\}$ be a sequence of functions $f_n : X \to Y$.

(a) Define what it means for the sequence $\{f_n\}$ to *converge uniformly* to a function $f : X \to Y$.

(b) Prove that if each f_n is bounded and $\{f_n\}$ converges uniformly to f, then f is also bounded. (Recall: a function $f : X \to Y$ is *bounded* if there is a constant $M \in \mathbb{R}_{\geq 0}$ such that if $x, x' \in X$ then $d_Y(f(x), f(x')) \leq M$.)

(c) Define $f_n : [0,1] \to \mathbb{R}$ for each $n \in \mathbb{Z}_{>0}$ by

$$f_n(x) = \frac{nx^2}{1+nx} \quad \text{for } x \in [0,1].$$

Find the pointwise limit f of the sequence $\{f_n\}$, and determine whether the sequence converges uniformly to f.

Question 7. Let $p \in \mathbb{R}_{>1}$. Let $e_i = (0,0,\ldots,0,1,0,0,\ldots)$ with 1 in the ith entry. Show that $\{e_1, e_2, e_3, \ldots\}$ is a Schauder basis of ℓ^p.

Question 8. Let $X = [0, 2\pi)$ and $Y = S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Let $f\colon [0,2\pi) \to S^1$ be given by

$$f(x) = (\cos x, \sin x).$$

(a) Show that f is continuous.

(b) Show that f is a bijection.

(c) Show that $f^{-1}\colon S^1 \to [0, 2\pi)$ is not continuous.

(d) Why does this not contradict the following statement: *Let X and Y be topological spaces and let $f\colon X \to Y$ be a continuous function. Assume f is a bijection, X is compact and Y is Hausdorff. Then the inverse function $f^{-1}\colon Y \to X$ is continuous.*