22 Problem list: Closures, continuity and limits

22.1 Neighborhoods

1. (Neighborhoods and neighborhood filters) Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$. A *neighborhood* of x is a subset N of X such that

$$\text{there exists } U \in \mathcal{T} \text{ such that } x \in U \text{ and } U \subseteq N.$$

The *neighborhood filter* of x is

$$\mathcal{N}(x) = \{\text{neighborhoods } N \text{ of } x\}.$$

Show that the collections $\mathcal{N}(x)$, for $x \in X$, satisfy

(a) If $A \subseteq X$ and there exists $N \in \mathcal{N}(x)$ such that $A \supseteq N$ then $A \in \mathcal{N}(x)$,
(b) If $\ell \in \mathbb{Z}_{>0}$ and $N_1, N_2, \ldots, N_\ell \in \mathcal{N}(x)$ then $N_1 \cap N_2 \cap \cdots \cap N_\ell \in \mathcal{N}(x)$,
(c) If $N \in \mathcal{N}(x)$ then $x \in N$,
(d) If $N \in \mathcal{N}(x)$ then there exists $W \in \mathcal{N}(x)$ such that if $y \in W$ then $N \in \mathcal{N}(y)$.

2. (Determining a topological space from neighborhoods) Let X be a set with a collection $\mathcal{N}(x)$ of subsets of X for each $x \in X$, which satisfy

(a) If $x \in X$ then $X \in \mathcal{N}(x)$,
(b) If $A \subseteq X$ and there exists $N \in \mathcal{N}(x)$ such that $A \supseteq N$ then $A \in \mathcal{N}(x)$,
(c) If $\ell \in \mathbb{Z}_{>0}$ and $N_1, N_2, \ldots, N_\ell \in \mathcal{N}(x)$ then $N_1 \cap N_2 \cap \cdots \cap N_\ell \in \mathcal{N}(x)$,
(d) If $N \in \mathcal{N}(x)$ then $x \in N$,
(e) If $N \in \mathcal{N}(x)$ then there exists $W \in \mathcal{N}(x)$ such that if $y \in W$ then $N \in \mathcal{N}(y)$.

Let

$$\mathcal{T} = \{A \subseteq X \mid \text{if } x \in A \text{ then } A \in \mathcal{N}(x)\}.$$

Show that

(a) Show that $\mathcal{T}$ is a topology on X.
(b) Show that the $\mathcal{N}(x)$, for $x \in X$, are the neighborhood filters for the topology $\mathcal{T}$.
(c) Show that $\mathcal{T}$ is unique topology on X such that $\mathcal{N}(x)$ for $x \in X$ are the neighborhood filters for $\mathcal{T}$.

3. (neighborhood filters of the uniform space topology) Let $(X, \mathcal{X})$ be a uniform space. Show that the uniform space topology on X is the unique topology such that

$$\text{if } x \in X \text{ then} \qquad \mathcal{N}(x) = \{B_V(x) \mid V \in \mathcal{X}\} \text{ is the neighborhood filter of } x.$$

4. (union generating set of a topology) Let $(X, \mathcal{T})$ be a topological space.

A *union generating set*, or *base, of* $\mathcal{T}$ is a collection $\mathcal{B}$ of subsets of X such that

$$\mathcal{T} = \{\text{unions of sets in } \mathcal{B}\}.$$

Show that $\mathcal{B}$ is a base of the topology $\mathcal{T}$ if and only if $\mathcal{B}$ satisfies

(a) (intersection covering) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then

$$\text{there exists } B \in \mathcal{B} \text{ such that} \quad x \in B \text{ and } B \subseteq B_1 \cap B_2.$$

(b) (cover) $\bigcup_{B \in \mathcal{B}} B = X$.

5. (inclusion generating set of the neighborhood filter) Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$ and let $\mathcal{N}(x)$ be the neighborhood filter of x. An *inclusion generating set for* $\mathcal{N}(x)$, or *fundamental system of neighborhoods of* x is a set $\mathcal{B}(x)$ of neighborhoods of x such that

$$\mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } B \in \mathcal{B}(x) \text{ such that } N \supseteq B\}.$$

Show that $\mathcal{B}$ is a union generating set of the topology $\mathcal{T}$ if and only if $\mathcal{B}$ satisfies

$$\text{if } x \in X \quad \text{then} \quad \mathcal{B}(x) = \{B \in \mathcal{B} \mid x \in B\}$$

is an inclusion generating set of $\mathcal{N}(x)$.

6. (The metric space topology) Let (X, d) be a metric space. Show that

$$\mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \mathbb{R}_{>0}, x \in X\}$$

is a base of the metric space topology on X.

7. (The product topology) Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. Show that

$$\mathcal{B} = \{U \times V \mid U \in \mathcal{T} \text{ and } V \in \mathcal{U}\}$$

is a base of the product topology on X.

22.2 Continuous and uniformly continuous functions

1. (the epsilon-delta version of continuity) Let (X, d) and (Y, ρ) be metric spaces and let $f\colon X \to Y$ be a function. Show that $f\colon X \to Y$ is continuous if and only if f satisfies

$$\begin{aligned}&\text{if } \epsilon \in \mathbb{R}_{>0} \text{ and } x \in X \text{ then there exists } \delta \in \mathbb{R}_{>0} \text{ such that}\\ &\qquad \text{if } y \in X \text{ and } d(x,y) < \delta \text{ then } \rho(f(x), f(y)) < \epsilon.\end{aligned}$$

2. (the epsilon-delta version of uniform continuity) Let (X, d) and (Y, ρ) be metric spaces and let $f\colon X \to Y$ be a function. Show that $f\colon X \to Y$ is uniformly continuous if and only if f satisfies

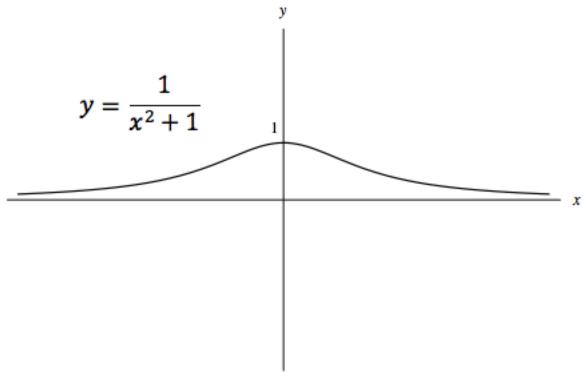
$$\begin{aligned}&\text{if } \epsilon \in \mathbb{R}_{>0} \text{ then there exists } \delta \in \mathbb{R}_{>0} \text{ such that}\\ &\qquad \text{if } x, y \in X \text{ and } d(x,y) < \delta \text{ then } \rho(f(x), f(y)) < \epsilon.\end{aligned}$$

3. (Uniformly continuous functions are continuous) Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be uniform spaces and let $f: X \to Y$ be a uniformly continuous function. Show that $f: X \to Y$ is continuous (with respect to the uniform space topology on X and Y).

4. (continuous functions are not uniformly necessarily continuous) Let $X = \mathbb{R}$ with metric given by $d(x, y) = |x - y|$.

   (a) Show the function $g: \mathbb{R} \to \mathbb{R}$ given by

   $$g(x) = \frac{1}{1+x^2} \qquad \text{is uniformly continuous.}$$

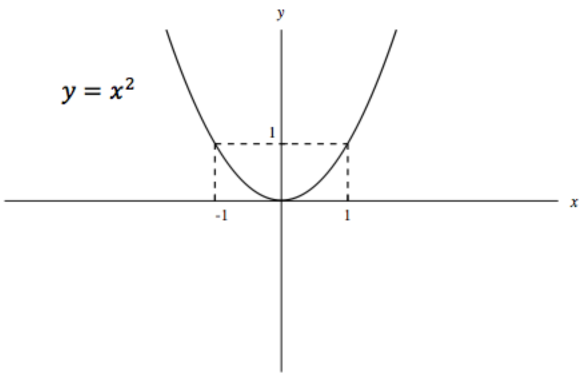


   (b) Show the function $g: \mathbb{R} \to \mathbb{R}$ given by

   $$g(x) = \frac{x}{1+x^2} \qquad \text{is uniformly continuous.}$$

   GRAPH THIS FUNCTION.

   (c) Show that the function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous but not uniformly continuous.



5. (continuous is the same as continuous at each point) Let X and Y be topological spaces and let $f: X \to Y$ be a function. Show that f is continuous if and only if

   $$f \text{ satisfies:} \quad \text{if } a \in X \quad \text{then} \quad f \text{ is continuous at } a.$$

6. (composition of continuous functions is continuous) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous functions. Show that $g \circ f$ is continuous.

7. (composition of uniformly continuous functions is uniformly continuous) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be uniformly continuous functions. Show that $g \circ f$ is uniformly continuous.

22.3 Sequences of functions

1. (sequences of functions) Let (X, d) and (C, ρ) be metric spaces. Let

$$F = \{\text{functions } f\colon X \to C\} \qquad \text{and define} \quad d_\infty\colon F \times F \to \mathbb{R}_{\geq 0} \cup \{\infty\} \quad \text{by}$$

$$d_\infty(f, g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$

(Warning d_∞ is not quite a metric since its target is not $\mathbb{R}_{\geq 0}$.) Let

$$(f_1, f_2, \ldots) \text{ be a sequence in } F \quad \text{and let} \quad f\colon X \to C$$

be a function.

The sequence $(f_1, f_2, \ldots)$ in F *converges pointwise to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

$$\begin{aligned}&\text{if } x \in X \text{ and } \epsilon \in \mathbb{R}_{>0} \text{ then there exists } n \in \mathbb{Z}_{>0} \text{ such that}\\ &\qquad \text{if } n \in \mathbb{Z}_{\geq N} \text{ then } d(f_n(x), f(x)) < \epsilon.\end{aligned}$$

The sequence $(f_1, f_2, \ldots)$ in F *converges uniformly to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

$$\begin{aligned}&\text{if } \epsilon \in \mathbb{R}_{>0} \text{ then there exists } n \in \mathbb{Z}_{>0} \text{ such that}\\ &\qquad \text{if } x \in X \text{ and } n \in \mathbb{Z}_{\geq N} \text{ then } d(f_n(x), f(x)) < \epsilon.\end{aligned}$$

(a) Show that $(f_1, f_2, \ldots)$ converges pointwise to f if and only if $(f_1, f_2, \ldots)$ satisfies

$$\text{if } x \in X \quad \text{then} \quad \lim_{n\to\infty} d(f_n(x), f(x)) = 0.$$

(b) Show that $(f_1, f_2, \ldots)$ converges uniformly to f if and only if $(f_1, f_2, \ldots)$ satisfies

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0.$$

2. (uniform convergence implies pointwise convergence) Let (X, d) and (C, ρ) be metric spaces. Let

$$F = \{\text{functions } f\colon X \to C\} \qquad \text{and define} \quad d_\infty\colon F \times F \to \mathbb{R}_{\geq 0} \cup \{\infty\} \quad \text{by}$$

$$d_\infty(f, g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$

(Warning d_∞ is not quite a metric since its target is not $\mathbb{R}_{\geq 0}$.) Let

$$(f_1, f_2, \ldots) \text{ be a sequence in } F \quad \text{and let} \quad f\colon X \to C$$

be a function.

The sequence $(f_1, f_2, \ldots)$ in F *converges pointwise to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

$$\text{if } x \in X \quad \text{then} \quad \lim_{n\to\infty} d(f_n(x), f(x)) = 0.$$

The sequence $(f_1, f_2, \ldots)$ in F *converges uniformly to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0.$$

Show that if $(f_1, f_2, \ldots)$ converges uniformly to f then $(f_1, f_2, \ldots)$ converges pointwise to f.

3. (pointwise convergence does not imply uniform convergence) Let (X, d) and (C, ρ) be metric spaces. Let

$$F = \{\text{functions } f\colon X \to C\}, \qquad (f_1, f_2, \ldots) \text{ a sequence in } F$$

and let $f\colon X \to C$ be a function.

(a) Show that if $(f_1, f_2, \ldots)$ converges uniformly to f then $(f_1, f_2, \ldots)$ converges pointwise to f.

(b) Let $X = C = \mathbb{R}_{[0,1]} = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$ with metric given by $d(x, y) = \rho(x, y) = |x - y|$. For $n \in \mathbb{Z}_{>0}$ let

$$f_n\colon \begin{matrix} \mathbb{R}_{[0,1]} & \to & \mathbb{R}_{[0,1]} \\ x & \mapsto & x^n \end{matrix} \qquad \text{and let } f\colon \mathbb{R}_{[0,1]} \to \mathbb{R}_{[0,1]}$$

be given by

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1, \\ 1, & \text{if } x = 1. \end{cases}$$

Show that $(f_1, f_2, \ldots)$ converges pointwise to f but does not converge uniformly to f.

GRAPH f_1, f_2, f_3, f_4 AND f

4. (uniformly convergent sequences of continuous functions have continuous limits) Let (X, d) and (C, ρ) be metric spaces. Let

$$F = \{\text{functions } f\colon X \to C\} \qquad \text{and define} \quad d_\infty\colon F \times F \to \mathbb{R}_{\ge 0} \cup \{\infty\} \quad \text{by}$$

$$d_\infty(f, g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$

(Warning d_∞ is not quite a metric since its target is not $\mathbb{R}_{\ge 0}$.) Let

$$(f_1, f_2, \ldots) \text{ be a sequence in } F \quad \text{and let} \quad f\colon X \to C$$

be a function.

The sequence $(f_1, f_2, \ldots)$ in F *converges uniformly to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0.$$

Show that if $f_1, f_2, \ldots$ are all continuous and $(f_1, f_2, \ldots)$ converges uniformly to f then f is continuous.

5. (the pointwise limit of continuous functions is not necessarily continuous) Let (X, d) and (C, ρ) be metric spaces. Let

$$F = \{\text{functions } f\colon X \to C\}, \qquad (f_1, f_2, \dots) \text{ a sequence in } F,$$

and let $f\colon X \to C$ be a function.

The sequence $(f_1, f_2, \ldots)$ in F *converges pointwise to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

$$\text{if } x \in X \quad \text{then} \quad \lim_{n\to\infty} d(f_n(x), f(x)) = 0.$$

Show that if $f_1, f_2, \ldots$ are all continuous and $(f_1, f_2, \ldots)$ converges pointwise to f then f is not necessarily continuous.

22.4 norms and metrics are continuous

1. (coordinate functions of a metric are continuous) Let $\mathbb{R}_{\ge 0}$ have the metric given by $d(x, y) = |x - y|$. Let X be a set and let $d\colon X \times X \to \mathbb{R}_{\ge 0}$ be a metric on X. Let $x \in X$. Show that the function

$$d_x\colon X \to \mathbb{R}_{\ge 0}, \quad \text{given by} \quad d_x(y) = d(x, y), \qquad \text{is continuous.}$$

2. (a metric is continuous) Let $\mathbb{R}_{\ge 0}$ have the metric given by $d(x, y) = |x - y|$. Let X be a set and let $d\colon X \times X \to \mathbb{R}_{\ge 0}$ be a metric on X. Using the metric space topology on X and the product topology on $X \times X$ show that

$$d\colon X \times X \to \mathbb{R}_{\ge 0}, \qquad \text{is continuous.}$$

3. (a norm is continuous) Let $\mathbb{R}_{\ge 0}$ have the metric given by $d(x, y) = |x - y|$. Let $(V, \|\ \|)$ be a normed vector space. Using the metric on V given by $d(x, y) = \|x - y\|$ and the metric space topology show that

$$\|\ \|\colon V \to \mathbb{R}_{\ge 0}, \qquad \text{is continuous.}$$

22.5 The Cantor set

1. (The Cantor set) Let $A = [0, 1] = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$ and remove the middle third of A to get

$$A_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]. \quad \text{DRAW A PICTURE OF } A_1$$

Now remove the middle third of each of the 2 components of A_1 to get

$$A_2 = [1, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1]. \qquad \text{DRAW A PICTURE OF } A_2$$

Then remove the middle third of each of the 4 components of A_2 to get

$$A_3 = [1, \tfrac{1}{27}] \cup [\tfrac{2}{27}, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{7}{27}] \cup [\tfrac{8}{27}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{19}{27}] \cup [\tfrac{20}{27}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, \tfrac{25}{27}] \cup [\tfrac{26}{27}, 1].$$

DRAW A PICTURE OF A_3

The *Cantor set* C is the subset of $[0,1]$ obtained by continuing this process,

$$C = \left((\tfrac{1}{3},\tfrac{2}{3}) \cup (\tfrac{1}{9},\tfrac{2}{9}) \cup (\tfrac{7}{9},\tfrac{8}{9}) \cup (\tfrac{1}{27},\tfrac{2}{27}) \cup (\tfrac{7}{27},\tfrac{8}{27}) \cup (\tfrac{19}{27},\tfrac{20}{27}) \cup (\tfrac{25}{27},\tfrac{26}{27}) \cup \cdots \right)^c,$$

where the complement is taken in $[0,1]$. (See [Bou] Top. Ch. IV §2 no. 5].)

Show that

(a) C is a closed subset of $[0,1]$.
(b) C is a nowhere dense subset of $[0,1]$.
(c) C is compact.
(d) C is totally disconnected.
(e) C has Lebesgue measure 0.
(f) $C = \left\{ a_1\left(\frac{1}{3}\right) + a_2\left(\frac{1}{3}\right)^2 + \cdots \mid a_1, a_2, \ldots \in \{0,2\} \right\}$.
(f) $\mathrm{Card}(C) = \mathrm{Card}(\mathbb{R})$.

22.6 Closed sets, closures, interiors and boundaries

1. (closed is not the same as not open) Let $X = \mathbb{R}$, $Y = \mathbb{R}_{(0,1)} = \{x \in \mathbb{R} \mid 0 < x < 1\}$ and $Z = \mathbb{R}_{[0,1]} = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$ all with metric $d(x,y) = |x-y|$.

   (a) Show that $(0,1]$ is not open in X and not closed in X.
   (b) Show that $(0,1)$ is open in X and not closed in X.
   (c) Show that $[0,1]$ is closed in X and not open in X.
   (d) Show that $\mathbb{R}$ is open in X and closed in X.
   (e) Show that $(0,1)$ is closed in Y and not closed in X.
   (f) Show that $[0,1]$ is open in Z and not open in X.
   (g) Show that $\mathbb{R}$ is closed and open in $\mathbb{R}$.
   (h) Show that $\mathbb{R}$ is closed and not open in $\mathbb{R}^2$.
   (j) Show that the Cantor set is closed in $[0,1] = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$.

2. (boundaries, dense sets and nowhere dense sets) Let $(X, \mathcal{T})$ be a topological space. Let $E \subseteq X$.

   The *boundary* of E is $\partial E = \overline{E} \cap \overline{E^c}$.
   The set E is *dense in* X if $\overline{E} = X$.
   The set E is *nowhere dense in* X if $(\overline{E})^\circ = \emptyset$.

   Show that

   (a) $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{Q}^\circ = \emptyset$.
   (b) $(0,1]$ is dense in $[0,1]$.
   (c) The boundary of $\mathbb{Q}$ in $\mathbb{R}$ is $\mathbb{R}$.
   (d) The boundary of $(0,1]$ in $\mathbb{R}$ is $\{0,1\}$. DRAW A PICTURE of $(0,1]$ and $\{0,1\}$.
   (e) $\mathbb{Z}_{>0}$ and $\mathbb{Z}$ are nowhere dense in $\mathbb{R}$.
   (f) $\mathbb{R}$ is nowhere dense in $\mathbb{R}^2$.
   (g) The Cantor set is nowhere dense in $[0,1]$.

3. (closure of the open ball of radius 1 is not always distance ≤ 1) Let (X, d) be a metric space. The *ball of radius ϵ centered at x* is

$$B_\epsilon(x) = \{y \in X \mid d(y, x) < \epsilon\}.$$

For a subset $A \subseteq X$ let $\overline{A}$ be the closure of A in X, in the metric space topology.

(a) Show that if $X = \mathbb{Z}$ with metric given by $d(x, y) = |x - y|$ then

$$\overline{B_1(0)} \neq \{y \in X \mid d(x, y) \le 1\}.$$

(b) Show that if $X = \mathbb{R}$ with metric given by $d(x, y) = |x - y|$ then

$$\overline{B_1(0)} = \{y \in X \mid d(x, y) \le 1\}.$$

(c) Let $X = \mathbb{R}^n$ with norm given by $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ for $x = (x_1, x_2, \ldots, x_n)$ and with metric given by $d(x, y) = \|x - y\|$ then

$$\overline{B_1(0)} = \{y \in X \mid d(x, y) \le 1\}.$$

4. (Closed sets in X) Let $(X, \mathcal{T})$ be a topological space. A *closed set in X* is a subset C of X such that the complement of C is a open set in X, i.e.

$$C^c = \{x \in X \mid x \notin C\} \qquad \text{is an open set in } X.$$

Show that $\mathcal{C} = \{C \subseteq X \mid C \text{ is a closed set}\}$ satisfies

(a) $\emptyset \in \mathcal{C}$ and $X \in \mathcal{C}$,
(b) If $\mathcal{S} \subseteq \mathcal{C}$ then $\left(\bigcap_{C \in \mathcal{S}} C\right) \in \mathcal{C}$,
(c) If $\ell \in \mathbb{Z}_{>0}$ and $C_1, C_2, \ldots, C_\ell \in \mathcal{C}$ then $C_1 \cup C_2 \cup \cdots C_\ell \in \mathcal{C}$.

5. (Determining a topological space from closed sets) Let X be a set and let $\mathcal{C}$ be a collection of subsets of X which satisfies

(a) $\emptyset \in \mathcal{C}$ and $X \in \mathcal{C}$,
(b) If $\mathcal{S} \subseteq \mathcal{C}$ then $\left(\bigcap_{C \in \mathcal{S}} C\right) \in \mathcal{C}$,
(c) If $\ell \in \mathbb{Z}_{>0}$ and $C_1, C_2, \ldots, C_\ell \in \mathcal{C}$ then $C_1 \cup C_2 \cup \cdots C_\ell \in \mathcal{C}$.

Let

$$\mathcal{T} = \{U \subseteq X \mid U^c \in \mathcal{C}\}.$$

Show that $\mathcal{T}$ is a topology on X.

6. (Interiors) Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$. The *interior* of E is the subset E° of X such that

(a) E° is open and $E^\circ \subseteq E$,
(b) If U is open and $U \subseteq E$ then $U \subseteq E^\circ$.

Show that E° exists and is unique.

7. (Interiors and interior points) Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$. The *interior* of E is the subset E° of X such that

   (a) E° is open and $E^\circ \subseteq E$,
   (b) If U is open and $U \subseteq E$ then $U \subseteq E^\circ$.

   An *interior point of* E is a element $x \in X$ such that there exists a neighborhood N of x such that $N \subseteq E$.

   Show that the interior of E is the set of interior points of E.

8. (Closures) Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$. The *closure of* E is the subset $\overline{E}$ of X such that

   (a) $\overline{E}$ is closed and $E \subseteq \overline{E}$,
   (b) If C is closed and $E \subseteq C$ then $\overline{E} \subseteq C$.

   Show that $\overline{E}$ exists and is unique.

9. (Interiors, closures and complements) Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$.

   (a) Show that $\overline{E^c} = (E^\circ)^c$, by using the definition of closure.
   (b) Show that $(E^c)^\circ = (\overline{E})^c$, by taking complements and using (a).

10. (Closures and close points) Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$. A *close point* to E is an element $x \in X$ such that if N is a neighborhood of x then $N \cap E \neq \emptyset$.

    (a) Let C be the set of close points to E and show that $C^c = (E^c)^\circ$.
    (b) Show that the closure of E is the set of close points of E.

22.7 Dense and nowhere dense sets

1. (boundaries, dense sets and nowhere dense sets) Let $(X, \mathcal{T})$ be a topological space. Let $E \subseteq X$.

   The *boundary* of E is $\partial E = \overline{E} \cap \overline{E^c}$.
   The set E is *dense in* X if $\overline{E} = X$.
   The set E is *nowhere dense in* X if $(\overline{E})^\circ = \emptyset$.

   Show that

   (a) $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{Q}^\circ = \emptyset$.
   (b) $(0, 1]$ is dense in $[0, 1]$.
   (c) The boundary of $\mathbb{Q}$ in $\mathbb{R}$ is $\mathbb{R}$.
   (d) The boundary of $(0, 1]$ in $\mathbb{R}$ is $\{0, 1\}$. PICTURE
   (e) $\mathbb{Z}_{>0}$ and $\mathbb{Z}$ are nowhere dense in $\mathbb{R}$.
   (f) $\mathbb{R}$ is nowhere dense in $\mathbb{R}^2$.
   (g) The Cantor set is nowhere dense in $[0, 1]$.

2. (intersection of two open dense sets is open and dense) Let (X, d) be a metric space and let $U \subseteq X$ and $V \subseteq X$. Show that if U and V are open and dense in X then $U \cap V$ is open and dense in X.

3. (intersection of two dense sets is not necessarily dense) Let $X = \mathbb{R}$ with the usual metric and let $U = \mathbb{Q}$ and $V = \mathbb{Q}^c$. Show that U and V are dense in $\mathbb{Q}$ and $U \cap V = \emptyset$.

4. (a sequence of open dense sets with empty intersection) Let $X = Q$ with the usual metric and let $\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}$ be an enumeration of $\mathbb{Q}$. For $n \in \mathbb{Z}_{>0}$ let $Q_n = \mathbb{Q} - \{q_n\}$.
   (a) Show that if $n \in \mathbb{Z}_{>0}$ then Q_n is open and dense in $\mathbb{Q}$.
   (b) Show that $\bigcap_{n \in \mathbb{Z}_{>0}} Q_n = \emptyset$.

5. (Baire category theorem, open dense version) Let (X, d) be a complete metric space and let $U_1, U_2, U_3, \ldots$ be a sequence of open and dense subsets of X. Show that $\bigcap_{n \in \mathbb{Z}_{>0}} U_n$ is dense in X.

6. (Baire category theorem, nowhere dense version) Let (X, d) be a complete metric space and let $F_1, F_2, F_3, \ldots$ be a sequence of nowhere dense subsets of X. Show that $\bigcup_{n \in \mathbb{Z}_{>0}} F_n$ has empty interior.

7. (Uniform boundedness) Let (X, d) be a complete metric space and let $f_1, f_2, f_3, \ldots$ be a sequence of

$$\text{continuous functions} \qquad f_n \colon X \to \mathbb{R}, \quad \text{for } n \in \mathbb{Z}_{>0}.$$

   Assume that

$$\text{if } x \in X \text{ then } \{f_1(x), f_2(x), \ldots\} \text{ is bounded in } X.$$

   Show that there exists an open set $U \subseteq X$ and $M \in \mathbb{R}_{>0}$ such that

$$\text{if } x \in U \text{ and } n \in \mathbb{Z}_{>0} \quad \text{then} \quad |f_n(x)| \le M.$$

8. Show that $\mathbb{R}$, with the standard topology, cannot be written as a countable union of nowhere dense sets.

9. Let X be a complete normed vector space over $\mathbb{R}$. A **sphere** in X is a set

$$S(a, r) = \{x \in X : d(x, a) = \|x - a\| = r\}$$

   where $a \in X$ and $r > 0$.
   (a) Show that each sphere in X is nowhere dense.
   (b) Show that there is no sequence of spheres $\{S_n\}$ in X whose union is X.
   (c) Give a geometric interpretation of the result in (b) when $X = \mathbb{R}^2$ with the Euclidean norm.
   (d) Show that the result of (b) does not hold in every complete metric space X.

22.8 Connected and path connected sets

1. (continuous images of connected sets are connected and continuous images of compact sets are compact) Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$. The set E is *connected* if there do not exist open sets A and B in X $(A, B \in \mathcal{T})$ with

$$A \cap E \neq \emptyset \quad \text{and} \quad B \cap E \neq \emptyset \quad \text{and} \quad A \cup B \supseteq E \quad \text{and} \quad (A \cap B) \cap E = \emptyset.$$

The set E is *compact* if E satisfies

$$\begin{gathered}\text{if } \mathcal{S} \subseteq \mathcal{T} \text{ and } E \subseteq \Big(\bigcup_{U \in \mathcal{S}} U\Big) \quad \text{then there exists} \\ \ell \in \mathbb{Z}_{>0} \text{ and } U_1, U_2, \ldots, U_\ell \in \mathcal{S} \text{ such that } E \subseteq U_1 \cup U_2 \cup \cdots \cup U_\ell.\end{gathered}$$

Let $f \colon X \to Y$ be a continuous function and let $E \subseteq X$. Show that

(a) If E is connected then $f(E)$ is connected,
(b) If E is compact then $f(E)$ is compact.

2. (characterizing connectedness via the subspace topology) Let $(X, \mathcal{T})$ be a topological space. A *connected set* is a subset $E \subseteq X$ such that there do not exist open sets A and B in X $(A, B \in \mathcal{T})$ with

$$A \cap E \neq \emptyset \quad \text{and} \quad B \cap E \neq \emptyset \quad \text{and} \quad A \cup B \supseteq E \quad \text{and} \quad (A \cap B) \cap E = \emptyset.$$

Let $\mathcal{T}_E$ be the subspace topology on E. Show that E is a connected set if and only if there do not exist open sets U and V in E $(U, V \in \mathcal{T}_E)$ with

$$U \neq \emptyset \quad \text{and} \quad V \neq \emptyset \quad \text{and} \quad U \cup V = E \quad \text{and} \quad U \cap V = \emptyset.$$

3. (closures of connected sets are connected) Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ be connected. Show that $\overline{A}$ is connected.

4. (connected subsets of $\mathbb{R}$ are intervals) Let $A \subseteq \mathbb{R}$, where the metric on $\mathbb{R}$ is given by $d(x, y) = |x - y|$. Show that

$$A \text{ is connected} \quad \text{if and only if} \quad A \text{ is an interval,}$$

i.e. A is connected if and only if there exist $a, b \in \mathbb{R} \cup \{\infty, -\infty\}$ such that $A = (a, b)$ or $A = [a, b)$ or $A = (a, b]$ or $A = [a, b]$.

5. (connected components of a topological space) Let $(X, \mathcal{T})$ be a topological space. Define a relation on X by

$$x \sim y \qquad \text{if there exists a connected set } E \subseteq X \text{ such that } x \in E \text{ and } y \in E.$$

Show that $\sim$ is an equivalence relation on X. The *connected components of* X are the equivalence classes with respect to the relation $\sim$. Show that the connected component containing x is the set

$$C_x = \bigcup_{\substack{E \subseteq X \text{ connected} \\ x \in E}} E.$$

6. (the connected components of $\mathbb{Q}$) Let $X = \mathbb{Q}$ with the metric given by $d(x, y) = |x - y|$. Show that the connected components of $\mathbb{Q}$ are the one point sets $\{x\}$, $x \in \mathbb{Q}$.

7. (path connected implies connected) Let $[0, 1] = \{a \in \mathbb{R} \mid 0 \le a \le 1\}$ with metric given by $d(a_1, a_2) = |a_1 - a_2|$ and the metric space topology. Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$. The set E is *path connected* if E satisfies

$$\text{if } x, y \in E \quad \text{then there exists a continuous function } f\colon [0,1] \to E \text{ with } f(0) = x \text{ and } f(1) = y.$$

Show that if E is path connected then E is connected.

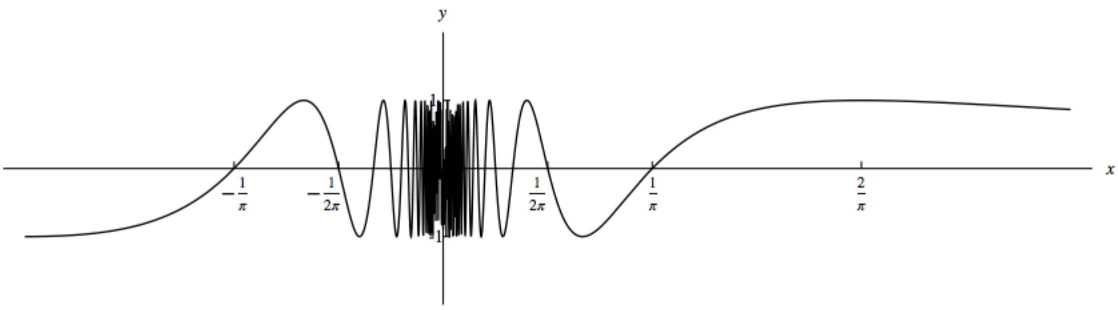
8. (connected does not imply path connected) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \end{cases}$$

and let

$$\Gamma = \{(x, f(x)) \in \mathbb{R}_{\ge 0} \times \mathbb{R} \mid x \in \mathbb{R}_{\ge 0}\} \qquad \text{be the graph of } f.$$

Show that $\Gamma \subseteq \mathbb{R} \times \mathbb{R}$ is connected but not path connected.



9. (continuous surjective functions $f\colon X \to \{0, 1\}$) Let $(X, \mathcal{T})$ be a topological space and let $\{0, 1\}$ have the discrete topology. Show that X is connected if and only if there does not exist a continuous surjective function $f\colon X \to \{0, 1\}$.

10. (totally disconnected sets) A topological space $(X, \mathcal{T})$ is *totally disconnected* if the connected components of X are the sets $\{x\}$, for $x \in X$.

   (a) Show that $\mathbb{Q}$ with the standard topology is totally disconnected.
   (b) Show that $\mathbb{Q}_p$ with the p-adic topology is totally disconnected.
   (c) Show that the Cantor set with the standard topology is totally disconnected.

22.9 First countable, second countable and separable spaces

Let $(X, \mathcal{T})$ be a topological space.

- $(X, \mathcal{T})$ is *first countable* if $\mathcal{N}(a)$ is countably generated for each $a \in X$,

i.e. $(X, \mathcal{T})$ is *first countable* if X satisfies: if $a \in X$ then

$$\begin{array}{l}\text{there exist } N_1, N_2, \ldots \in \mathcal{N}(a) \text{ such that} \\ \text{if } N \in \mathcal{N}(a) \quad \text{then} \quad \text{there exists } r \in \mathbb{Z}_{>0} \text{ such that } N \supseteq N_r.\end{array}$$

- $(X, \mathcal{T})$ is *second countable* if $\mathcal{T}$ is countably generated,

i.e. $(X, \mathcal{T})$ is *second countable* if X satisfies:

$$\begin{array}{l}\text{there exist } U_1, U_2, \ldots \in \mathcal{T} \text{ such that} \\ \text{if } U \in \mathcal{T} \quad \text{then} \quad \text{there exists } S \subseteq \mathbb{Z}_{>0} \text{ such that} \quad U = \bigcup_{s \in S} U_s.\end{array}$$

- $(X, \mathcal{T})$ is *separable* if it has a countable dense set,

i.e. $(X, \mathcal{T})$ is *separable* if X satisfies:

$$\text{there exist } x_1, x_2, \ldots \in X \text{ such that } \overline{\{x_1, x_2, \ldots\}} = X.$$

1. (Second countable implies first countable) Let $(X, \mathcal{T})$ be a topological space. Show that if $(X, \mathcal{T})$ is second countable then $(X, \mathcal{T})$ is first countable.

2. (Second countable implies separable) Let $(X, \mathcal{T})$ be a topological space. Show that if $(X, \mathcal{T})$ is second countable then $(X, \mathcal{T})$ is separable.

3. (separable does not imply second countable) Show that $\mathbb{R}$ with the topology $\mathcal{T} = \{\text{unions of } [a, b)\}$ is separable but not second countable.

4. (first countable does not imply second countable) Show that $\mathbb{R}$ with the discrete topology is first countable but not second countable.

5. (a topological space that is not first countable) Show that $\mathbb{R}$ with the topology $\mathcal{T} = \{U \subseteq \mathbb{R} \mid U^c \text{ is a finite set}\}$ is a topological space that is not first countable.

6. (metric spaces are first countable) Let (X, d) be a metric space. Show that X with the metric space topology is first countable.

7. (for metric spaces, second countable is equivalent to separable) Let (X, d) be a metric space with the metric space topology. Show that X is second countable if and only if X is separable.

8. (metric spaces are not always separable)

   (a) Show that $\mathbb{R}$ with the standard topology is separable.
   (b) Show that $\mathbb{R}$ with the discrete topology is not separable.
   (b) Show that $\mathbb{R}^n$ is separable.
   (c) Show that ℓ^1 is separable.
   (d) Let $p \in \mathbb{R}_{>1}$. Show that ℓ^p is separable.
   (e) Show that ℓ^∞ is not separable.

9. (closure and limits of sequences in first countable spaces) Let $(X, \mathcal{T})$ be a first countable topological space and let $A \subseteq X$. Then

$$\overline{A} = \{z \in X \mid \text{there exists a sequence } (a_1, a_2, \ldots) \text{ in } A \text{ such that } z = \lim_{n\to\infty} a_n\},$$

   where $\overline{A}$ is the closure of A in X.

10. (continuity and limits of sequences in first countable spaces) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces such that $(X, \mathcal{T}_X)$ is first countable. Let $f\colon X \to Y$ be a function. Then f is continuous if and only if f satisfies

$$\text{if } (x_1, x_2, \ldots) \text{ is a sequence in } X \text{ and } \lim_{n\to\infty} x_n \text{ exists} \quad \text{then} \quad f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n).$$

   This result says that, when $(X, \mathcal{T}_X)$ is first countable, f is continuous if and only if f commutes with $\lim_{n\to\infty}$.

11. (metric spaces with a countable dense set have a countable base) [BR, Ch. 2 Ex. 23] A metric space (X, d) is *separable* if it has a countable dense set.
   A *base* of a topological space $(X, \mathcal{T})$ is a subset $\mathcal{B}$ of $\mathcal{T}$ such that every open set of X is a union of elements of $\mathcal{B}$. Show that if X has a countable dense subset A then the open balls $B_\epsilon(a)$ for $\epsilon \in \mathbb{Q}$, $a \in A$ form a countable base of X (with the metric space topology). IS IT ENOUGH TO TAKE $B_\epsilon(a)$ with $\epsilon \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and $a \in A$????

12. (countable dense sets in topological spaces) [Bou, Ch. I §1 Ex. 7 and its footnotes] Consider the following four properties of a topological space $(X, \mathcal{T})$.

   (D_I) X has a countable base.
   (D_{II}) X has a countable dense set.
   (D_{III}) Every subset of X, all of whose points are isolated, is countable.
   (D_{IV}) Every set of mutually disjoint non-empty open subsets of X is countable.

   Show that $(D_I) \Rightarrow (D_{II})$, $(D_I) \Rightarrow (D_{III})$, $(D_{II}) \Rightarrow (D_{IV})$ $(D_{III}) \Rightarrow (D_{IV})$. MAKE A PICTURE THAT SHOwS THIS

   For $(D_{IV}) \not\Rightarrow (D_{III})$ and $(D_{IV}) \not\Rightarrow (D_{II})$ see [Bou, Top. Ch. I §8 Ex. 6b].
   For $(D_{II}) + (D_{III}) \not\Rightarrow (D_I)$, see [Bou, Top. Ch. IX §5 Ex. 16].
   For $(D_{II}) \not\Rightarrow (D_{III)})$ see [Bou, Top. Ch. I §9 Ex. 23].
   For $(D_{III}) \not\Rightarrow (D_{II)})$ see [Bou, Top. Ch. I §9 Ex. 23].

13. $(D_{IV}) \not\Rightarrow (D_{III})$: Let $A = \mathcal{P}(\mathbb{Z}_{>0})$, where $\mathcal{P}(X)$ denotes the set of subsets of X. Let $\{0,1\}$ have the discrete topology. Show that the product space $\{0,1\}^A$ satisfies (D_{IV}) and does not satisfy (D_{III}). (See [Bou, Top. Ch. I §4 Ex. 4b and c].)

14. $(D_{II}) \not\Rightarrow (D_{III})$: Let $A = \mathcal{P}(\mathbb{Z}_{>0}))$, where $\mathcal{P}(X)$ denotes the set of subsets of X. Let $\{0,1\}$ have the discrete topology. Show that the product space $\{0,1\}^A$ satisfies (D_{II}) and does not satisfy (D_{III}). (See [Bou, Top. Ch. I §4 Ex. 5b].)

15. $((D_{III}) \not\Rightarrow (D_{II}))$ Let $X_0 = [0,1]$ with the standard topology. Let $\mathcal{T}$ be the topology on $[0,1]$ generated by the open intervals in $[0,1]$ and the complements of countable sets in $[0,1]$. Show that $([0,1], \mathcal{T})$ satisfies (D_{III}) and does not satisfy (D_{II}). (See [Bou, Top. Ch. I §9 Ex. 23c].)

16. $((D_{II})) + (D_{III}) \not\Rightarrow (D_I))$ LOOK THIS UP IN THE NEW VERSION (See [Bou, Top. Ch. IX §5 Ex. 16].)

17. (countable dense sets in metric spaces) [Bou, Top. Ch. IX §2 no. 8 Proposition 12] and [Bou, Top. Ch. I §1 Ex. 7 footnote]. Let (X,d) be a metric space. Show that the following are equivalent.

- (D_I) X has a countable base.
- (D_{II}) X has a countable dense set.
- (D_{III}) Every subset of X, all of whose points are isolated, is countable.
- (D_{IV}) Every set of mutually disjoint non-empty open subsets of X is countable.

22.10 Additional sample exam questions

22.10.1 Open and closed sets and limits

1. Let X be a topological space and let $x \in X$. Consider the following definitions of "neighborhood of x":

   A *neighborhood* of x is a set $N \subseteq X$ such that $x \in N^\circ$.
   A *neighborhood* of x is a set $V \subseteq X$ such that there exists an open set U of X with $x \in U \subseteq V$.

   Show that these two definitions of "neighborhood of x" are equivalent.

2. Let $X = \mathbb{R}^2$. For $x = (x_1, x_2)$ and $y = (y_1, y_2) \in X$ define

$$d_M(x,y) = \begin{cases} |x_2 - y_2|, & \text{if } x_1 = y_1, \\ |x_1 - y_1| + |x_2| + |y_2|, & \text{if } x_1 \neq y_1. \end{cases}$$

   Also let $\|x\| - (x_1^2 + x_2^2)^{\frac{1}{2}}$ and define

$$d_K(x,y) = \begin{cases} \|x - y\|, & \text{if } x = ty \text{ for some } t \in \mathbb{R}; \\ \|x\| + \|y\| \quad , & \text{otherwise.} \end{cases}$$

   (Can you give reasonable interpretations of the metrics d_M and d_K?)

   Study the convergence of the sequence x_n in the spaces (X, d_M) and (X, d_K) if

(a) $x_n = (\frac{1}{n}, \frac{n}{n+1})$;
(b) $x_n = (\frac{n}{n+1}, \frac{n}{n+1})$;
(c) $x_n = (\frac{1}{n}, \sqrt{n+1} - \sqrt{n})$.

3. Let $\{x_n\}$ and $\{y_n\}$ be sequences in a metric space (X, d) such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Prove that $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

4. Let C be the circle in $\mathbb{R}^2$ with the centre at $(0, 1/2)$ and radius $1/2$. Let $X = C \setminus \{(0, 1)\}$. Define the function $f : \mathbb{R} \to X$ by defining $f(t)$ to be the point at which the line segment from $(t, 0)$ to $(0, 1)$ intersects X.

   (a) Show that $f : \mathbb{R} \to X$ and $f^{-1} : X \to \mathbb{R}$ are continuous.
   (b) Define for $s, t \in \mathbb{R}$
   $$\rho(s, t) = |f(s) - f(t)|$$
   where $|\ |$ is the standard norm in $\mathbb{R}^2$. Show that ρ defines a metric on $\mathbb{R}$ which is topologically equivalent to the standard metric on $\mathbb{R}$.

5. Let d and d' be topologically equivalent metrics on X. Show that

   (a) $A \subseteq X$ is closed in (X, d) if and only if A is closed in (X, d');
   (b) $A \subseteq X$ is open in (X, d) if and only if A is open in (X, d').

6. Let X be a metric space and let $x_1, x_2, \ldots$ be a sequence in X. Show that $\lim_{n\to\infty} x_n$ is unique, if it exists.

7. Let X be a topological space and let E be a subset of X. Let $x \in X$. Show that x is a close point of E if and only if there exists a sequence $x_1, x_2, \ldots$ of points in E such that $\lim_{n\to\infty} x_n = x$.

8. Let (X, d) be a metric space, let $A \subseteq X$ and let $\overline{A}$ be the closure of A in X. Show that
$$\overline{A} = \{z \in X \mid \text{there exists a sequence } (a_1, a_2, \ldots) \text{ in } A \text{ with } \lim_{n\to\infty} a_n = z\}.$$

9. Let X be a topological space and let E be a subset of X. Let E° be the interior of E. Show that E is open if and only if $E = E^\circ$.

10. Let X be a topological space and let E be a subset of X. Let E° be the interior of E. Show that E° is the set of interior points of E.

11. Let X be a topological space and let E be a subset of X. Let $\overline{E}$ be the closure of E. Show that E is closed if and only if $E = \overline{E}$.

12. Let (X, d) be a metric space and let $x \in X$ and $r \in \mathbb{R}_{>0}$. Show that

$$B_{\leq r}(x) = \{y \in X \mid d(x,y) \leq r\}$$

is a closed set in the metric space topology on X.

13. Give an example of a metric space (X, d) and a point $x \in X$ such that $B_{\leq 1}(x) \neq \overline{B_1(x)}$.

14. Let (X, d) be a metric space and let $x \in X$ and $r \in \mathbb{R}_{>0}$. Show that $\overline{B_r(x)} \subseteq B_{\leq r}(x, r)$.

15. Let X be a set with the discrete metric d. Show that every subset of X is both open and closed (in the metric space topology on X).

16. Let X be a topological space. Show that X is discrete if and only if the only convergent sequences are those which are eventually constant.

17. Let X be a set and let $\mathcal{C}$ be a collection of subsets of X. Show that $\mathcal{C}$ is the set of closed sets for a topology on X if and only if $\mathcal{C}$ satisfies

    (a) finite unions of elements of $\mathcal{C}$ are in $\mathcal{C}$,
    (b) Arbitrary intersections of elements of $\mathcal{C}$ are in $\mathcal{C}$, $\emptyset \in \mathcal{C}$ and $X \in \mathcal{C}$.

18. Let A be an open subset of a metric space (X, d).

    (a) Show, directly from the definition, that if $b \in A$ then $A \setminus \{b\}$ is open in X.
    (b) If B is a finite subset of A show, using (a) or otherwise, that $A \setminus B$ is open in X.
    (c) Deduce that every finite subset of X is closed in X.

19. Let X be a topological space, and let A be a subset of X.

    (a) Define the *closure* $\overline{A}$ of A. (Give a definition in terms of closed sets.)
    (b) Show that $x \in \overline{A}$ if and only if every open neighbourhood of x intersects A.
    (c) Using (b) or otherwise, show that if $f : X \to Y$ is a continuous map between topological spaces and $A \subseteq X$ then $f(\overline{A}) \subseteq \overline{f(A)}$.

20. Let $(X, \mathcal{T})$ be a topological space.

    (a) Define the interior A^0 of a subset $A \subseteq X$
    (b) Prove that $(A \cap B)^0 = A^0 \cap B^0$.
    (c) Define the closure $\overline{A}$ of $A \subseteq X$. Give an example of subsets A, B in the real line $\mathbb{R}$, with the usual Euclidean topology, which satisfy $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

21. Let (X, d) and $Y, d')$ be metric spaces and let $f, g : X \to Y$ be continuous.

    (a) Show that the set $\{x \in X : f(x) = g(x)\}$ is a closed subset of X.

(b) Show that if $f, g : X \to \mathbb{R}$ are continuous, then $f - g$ is continuous and $\{x \in X : f(x) < g(x)\}$ is open.

22. Let $(X, \mathcal{T})$ be a topological space. Let U be open in X and let A be closed in X. Show that $U \setminus A$ is open in X and $A \setminus U$ is closed in X.

23. Consider the set $X = [-1, 1]$ as a metric subspace of $\mathbb{R}$ with the standard metric. Let

    (a) $A = \{x \in X \mid 1/2 < |x| < 2\}$;
    (b) $B = \{x \in X \mid 1/2 < |x| \le 2\}$;
    (c) $C = \{x \in \mathbb{R} \mid 1/2 \le |x| < 1\}$;
    (d) $D = \{x \in \mathbb{R} \mid 1/2 \le |x| \le 1\}$;
    (e) $E = \{x \in \mathbb{R} \mid 0 < |x| \le 1 \text{ and } 1/x \notin \mathbb{Z}\}$.

    Classify the sets in (a)–(e) as open/closed in X and $\mathbb{R}$.

24. Consider $\mathbb{R}^2$ with the standard metric. Let

    (a) $A = \{(x, y) \mid -1 < x \le 1 \text{ and } -1 < y < 1\}$;
    (b) $B = \{(x, y) \mid xy = 0\}$;
    (c) $C = \{(x, y) \mid x \in \mathbb{Q}, y \in \mathbb{R}\}$;
    (d) $D = \{(x, y) \mid -1 < x < 1 \text{ and } y = 0\}$;
    (e) $E = \bigcup_{n=1}^{\infty} \{(x, y) \mid x = 1/n \text{ and } |y| \le n\}$.

    Sketch (if possible) and classify the sets in (a)–(e) as open/closed/neither in $\mathbb{R}^2$.

25. Find the interior, the closure and the boundary of each of the following subsets of $\mathbb{R}^2$ with the standard metric:

    (a) $A = \{(x, y)) \mid x > 0 \text{ and } y \neq 0\}$;
    (b) $B = \{(x, y) \mid x \in \mathbb{Z}_{>0}, y \in \mathbb{R}\}$;
    (c) $C = A \cup B$;
    (d) $D = \{(x, y) \mid x \text{ is rational}\}$;
    (e) $F = \{(x, y) \mid x \neq 0 \text{ and } y \le 1/x\}$.

26. Let A be a subset of a metric space X. Is the interior of A equal to the interior of the closure of A? Is the closure of the interior of A equal to the closure of A itself?

27. Consider a collection $\{A_i\}_{i \in I}$ of subsets of a metric space X. Show that

$$\bigcup_{i \in I} A_i^\circ \subseteq \left(\bigcup_{i \in I} A_i \right)^\circ \qquad \overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}$$

$$\left(\bigcap_{i \in I} A_i \right)^\circ \subseteq \bigcap_{i \in I} A_i^\circ \qquad \bigcup_{i \in I} \overline{A_i} \subseteq \overline{\bigcup_{i \in I} A_i}$$

28. Let (X, d) be a metric space. Show that if $A \subseteq X$, then

    (a) $\overline{A} = A \cup \partial A$.
    (b) $\partial A = \overline{A} \setminus A^\circ$ and $A^\circ = A \setminus \partial A$.
    (c) A is closed if and only if $\partial A = A \setminus A^\circ$.
    (d) A is open if and only if $\partial A = \overline{A} \setminus A$.

29. Let X and Y be metric spaces and A, B non-empty subsets of X and Y, respectively. Prove that

    (a) If $A \times B$ is an open subset of $X \times Y$, then A and B are open in X and Y, respectively.
    (b) If $A \times B$ is a closed subset of $X \times Y$, then A and B are closed in X and Y, respectively.

30. Let (X, d_X) and (Y, d_Y) be metric spaces and A, B are dense subsets of X and Y, respectively. Show that $A \times B$ is dense in $X \times Y$.

31. Let $(X_1, d_1), \ldots, (X_\ell, d_\ell)$ be metric spaces. Show that a sequence $\overline{x_n} = (x_n^{(1)}, \ldots, x_n^{(\ell)})$ in $X_1 \times \cdots \times X_\ell$ converges if and only if each of the sequences $x_n^{(i)}$ (in X_i) converges.

32. Let X be a topological space and let $A \subseteq X$. Show that if $x \in X$ satisfies

$$\text{if } r \in \mathbb{R}_{>0} \text{ then } B_r(x) \cap A \neq \emptyset \text{ and } B_r(x) \cap A^c \neq \emptyset \text{ then } x \in \partial A.$$

33. Let X be a topological space and let $A \subseteq X$. Show that ∂A is a closed subset of X.

34. Let $X = \mathbb{R}$ with the usual topology.

    (a) Determine (with proof) $\partial([0, 1])$.
    (b) Determine $\partial \mathbb{Q}$ (with proof, of course).

35. Let (X, d) be a metric space. Let $x \in X$. Show that $\{x\} \subseteq X$ is closed (in the metric space topology on X).

36. Let (X, d) be a metric space and let $x \in X$. Show that x is isolated if and only if there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(x) = \{x\}$.

37. Let $X = \mathbb{R}$ with the usual topology. Show that

    (a) $\mathbb{Z}_{>0}$ is a discrete set in $\mathbb{R}$.
    (b) $\{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\} \subseteq \mathbb{R}$ is a discrete set in $\mathbb{R}$.

38. In $\mathbb{R}$ with the usual topology give an example of

(a) a set $A \subseteq \mathbb{R}$ which is both open and closed,
(b) a set $B \subseteq \mathbb{R}$ which is open and not closed,
(c) a set $C \subseteq \mathbb{R}$ which is closed and not open,
(d) a set $D \subseteq \mathbb{R}$ which is not open and not closed.

39. Let $X = \mathbb{R}$ with the usual topology. Show that

(a) $[0, 1) \subseteq \mathbb{R}$ is not open and not closed,
(b) $\mathbb{Q} \subseteq \mathbb{R}$ is not open and not closed.

40. Let $X = \mathbb{R}$ with the usual topology.

(a) Show that $\mathbb{Q}$ is dense in $\mathbb{R}$.
(b) Show that $\mathbb{Q}^c$ is dense in $\mathbb{R}$.
(c) Show that $\mathbb{Z}_{>0}$ is nowhere dense in $\mathbb{R}$.
(d) Show that $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$.
(e) Show that $\mathbb{R}$ is nowhere dense in $\mathbb{R}^2$.

41. Let C be the Cantor set in $\mathbb{R}$, where $\mathbb{R}$ has the usual topology.

(a) Show that C is closed in $\mathbb{R}$.
(b) Show that C does not contain any interval in $\mathbb{R}$.
(c) Show that C has nonempty interior.
(d) Show that C is nowhere dense in $\mathbb{R}$.

22.10.2 Continuity

1. Let (X, d) and (Y, ρ) be metric spaces and let $f: X \to Y$ be a function. Let $a \in X$. Show that f is continuous at a if and only if f satisfies:

$$\text{if } \varepsilon \in \mathbb{R}_{>0} \text{ then there exists } \delta \in \mathbb{R}_{>0} \text{ such that if } x \in \text{ and } d(x, a) < \delta \text{ then } \rho(f(x), f(a)) < \varepsilon.$$

2. Let X and Y be topological spaces and let $f: X \to Y$ be a function. Show that f is continuous if and only if f satisfies:

$$\text{if } a \in X \text{ then } f \text{ is continuous at } a.$$

3. Let X and Y be metric spaces and let $f: X \to Y$ be a function. Let $a \in X$. Show that f is continuous at a if and only if f satisfies:

$$\text{if } \varepsilon \in \mathbb{R}_{>0} \text{ then there exists } \delta \in \mathbb{R}_{>0} \text{ such that } f(B_\delta(a)) \subseteq B_\varepsilon(f(a)).$$

4. Let X and Y be metric spaces and let $f\colon X \to Y$ be a function. Let $a \in X$. Show that f is continuous at a if and only if f satisfies

$$\text{if } x_1, x_2, \ldots \text{ is a sequence in } X \text{ and } \lim_{n\to\infty} x_n = x_0 \quad \text{then} \quad \lim_{n\to\infty} f(x_n) = f(x_0).$$

5. Let X and Y be metric spaces and let $f\colon X \to Y$ be a function. Let $a \in X$. Show that f is continuous at a if and only if f satisfies:

$$\text{if } x_1, x_2, \ldots \text{ is a convergent sequence in } X \text{ then } \lim_{n\to\infty} f(x_n) = f\big(\lim_{n\to\infty} x_n\big).$$

6. Let X and Y be topological spaces. Let $f\colon X \to Y$ be a function. Show that f is continuous if and only if f satisfies: if $F \subseteq Y$ is closed then $f^{-1}(F)$ is closed in X.

7. Let X, Y and Z be topological spaces and let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous functions. Show that $g \cdot f$ is a continuous function.

8. Let X, Y be topological spaces and let $f\colon X \to Y$ be a continuous function. Let $A \subseteq X$. Show that the restriction of f to A, $f|_A\colon A \to Y$ is continuous.

9. Let (X, d), (Y_1, ρ_1) and (Y_2, ρ_2) be metric spaces. Let $f\colon X \to Y_1$ and $g\colon X \to Y_2$ be functions. Define $h\colon X \to Y_1 \times Y_2$ by $h(x) = (f(x), g(x))$. Let $a \in X$. Show that h is continuous if and only if f and g are continuous at a.

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ & & \big\downarrow {\scriptstyle f\times g} \\ & & Y_1 \times Y_2 \end{array} \qquad\qquad \begin{array}{ccc} x & \mapsto & (x,x) \\ & & \downarrow \\ & & (f(x), g(x)) \end{array}$$

10. For a topological space X and a sequence $\vec{x} = (x_1, x_2, \ldots)$ in X write

$$y = \lim_{n\to\infty} x_n, \qquad \begin{array}{l} \text{if } y \text{ is a limit point of } \vec{x}\colon \mathbb{Z}_{>0} \to X \\ \text{with respect to the tail filter on } \mathbb{Z}_{>0}. \end{array}$$

(a) Let X and Y be topological spaces. Define what it means for a function $f\colon X \to Y$ to be continuous.
(b) Let X and Y be uniform spaces. Define what it means for a function $f\colon X \to Y$ to be uniformly continuous.
(c) Let X and Y be uniform spaces. Show that if $f\colon X \to Y$ uniformly continuous then $f\colon X \to Y$ is continuous.
(d) Let (X, d) and (Y, ρ) be metric spaces and let $f\colon X \to Y$ be a function. Show that $f\colon X \to Y$ is continuous if and only if f satisfies

$$\begin{array}{c} \text{if } \epsilon \in \mathbb{R}_{>0} \text{ and } x \in X \text{ then there exists } \delta \in \mathbb{R}_{>0} \text{ such that} \\ \text{if } y \in X \text{ and } d(x,y) < \delta \text{ then } \rho(f(x), f(y)) < \epsilon. \end{array}$$

(e) Let (X, d) and (Y, ρ) be metric spaces and let $f\colon X \to Y$ be a function. Show that $f\colon X \to Y$ is uniformly continuous if and only if f satisfies

$$\begin{array}{l}\text{if } \epsilon \in \mathbb{R}_{>0} \text{ then there exists } \delta \in \mathbb{R}_{>0} \text{ such that} \\ \quad \text{if } x, y \in X \text{ and } d(x, y) < \delta \text{ then } \rho(f(x), f(y)) < \epsilon.\end{array}$$

(f) Let (X, d) and (Y, ρ) be metric spaces and let $f\colon X \to Y$ be a function. Show that f is continuous if and only if f satisfies

$$\text{if } (x_1, x_2, \ldots) \text{ is a sequence in } X \text{ and } \lim_{n\to\infty} x_n \text{ exists} \quad \text{then} \quad f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n).$$

11. (Functions on $\mathbb{R}_{\geq 0}$)

(a) Carefully define continuous and uniformly continuous functions.
(a) Let $n \in \mathbb{Z}_{>0}$. Prove that the function $x^n\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous.
(b) Let $n \in \mathbb{Z}_{>1}$. Prove that the function $x^n\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is not uniformly continuous.
(b) Let $n \in \{0, 1\}$. Prove that the function $x^n\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is uniformly continuous.
(c) Prove that the function $e^x\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous.

12. Let $X = [0, 2\pi)$ and $Y = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Let $f\colon [0, 2\pi) \to S^1$ be given by

$$f(x) = (\cos x, \sin x).$$

(a) Show that f is continuous.
(b) Show that f is a bijection.
(c) Show that $f^{-1}\colon S^1 \to [0, 2\pi)$ is not continuous.
(d) Why does this not contradict the following statement: *Let X and Y be topological spaces and let $f\colon X \to Y$ be a continuous function. Assume f is a bijection, X is compact and Y is Hausdorff. Then the inverse function $f^{-1}\colon Y \to X$ is continuous.*

13. Let $X = \mathbb{R}_{\geq 0}$ with metric given by $d(x, y) = |x - y|$. Show that the function

$$\begin{array}{ccc}\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} & \to & \mathbb{R}_{\geq 0} \\ (x, y) & \mapsto & x + y\end{array} \quad \text{is uniformly continuous}$$

and the function

$$\begin{array}{ccc}\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} & \to & \mathbb{R}_{\geq 0} \\ (x, y) & \mapsto & xy\end{array} \qquad \text{is continuous but not uniformly continuous.}$$

14. Let (X, d), (Y_1, ρ_1) and (Y_2, ρ_2) be metric spaces. Let $f\colon X \to Y_1$ and $g\colon X \to Y_2$ be functions. Define

$$h\colon X \to Y_1 \times Y_2 \qquad \text{by} \qquad h(x) = (f(x), g(x)).$$

Show that h is continuous if and only if f and g are continuous.

15. Let X be a topological space and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be continuous functions.

(a) Show that $f+g$ is continuous.
(b) Show that $f\cdot g$ is continuous.
(a) Show that $f-g$ is continuous.
(d) Show that if g satisfies if $x\in X$ then $g(x)\neq 0$ then f/g is continuous.

16. Let (X,d) be a metric space. Show that $d\colon X\times X\to\mathbb{R}$ is continuous.

17. Let $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be given by

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & \text{if } (x,y)\neq(0,0),\\ 0, & \text{if } (x,y)=(0,0).\end{cases}$$

If $a\in\mathbb{R}$ let $\ell_a\colon\mathbb{R}\to\mathbb{R}$ be given by $\ell_a(y)=f(a,y)$. If $b\in\mathbb{R}$ let $r_b\colon\mathbb{R}\to\mathbb{R}$ be given by $r_b(x)=f(x,b)$.

(a) Let $a\in\mathbb{R}$. Show that $\ell_a\colon\mathbb{R}\to\mathbb{R}$ is continuous.
(b) Let $b\in\mathbb{R}$. Show that $r_b\colon\mathbb{R}\to\mathbb{R}$ is continuous.
(c) Show that f is not continuous at (0,0).

18. Give an example of metric spaces X, Y and Z and a function $f\colon X\times Y\to Z$ such that

(a) if $x\in X$ then $\begin{matrix}\ell_x\colon & Y & \to & Z\\ & y & \mapsto & f(x,y)\end{matrix}$ is continuous,
(b) if $y\in Y$ then $\begin{matrix}r_y\colon & X & \to & Z\\ & x & \mapsto & f(x,y)\end{matrix}$ is continuous, and
(c) $f\colon X\times Y\to Z$ is not continuous.

19. Let X be a topological space and let $A\subseteq X$ and $B\subseteq X$ be closed subsets of X such that $X=A\cup B$. Let Y be a topological space and let $f\colon A\to Y$ and $g\colon B\to Y$ be continuous functions such that if $x\in A\cap B$ then $f(x)=g(x)$. Define $h\colon X\to Y$ by

$$h(x)=\begin{cases}f(x), \text{if } x\in A,\\ g(x), \text{if } x\in B.\end{cases}$$

Show that $h\colon X\to Y$ is continuous.

20. Show that the function $f\colon\mathbb{R}\to\mathbb{R}$ given by

$$f(x)=\frac{x}{1+x^2}\qquad\text{is uniformly continuous.}$$

21. Show that the function $f\colon\mathbb{R}\to\mathbb{R}$ given by $f(x)=x^2$, is not uniformly continuous.

22. Let (X, d) and (Y, ρ) be metric spaces and let $f: X \to Y$ be a function. Show that if f is uniformly continuous then f is continuous.

23. Let $X = C[0, 1]$. Let
$$F: X \to \mathbb{R} \qquad \text{be defined by} \quad F(f) = f(0).$$
Let
$$d_\infty(f, g) = \sup\{|f(x) - g(x)| \mid x \in [0, 1]\} \qquad \text{and}$$
$$d_1(f, g) = \int_0^1 |f(x) - g(x)| dx.$$
Is F continuous when X is equipped with (a) the metric d_∞, (b) the metric d_1?

24. Let (X, d_X) and (Y, d_Y) be metric spaces. Show that $f: X \to Y$ is continuous if and only if f satisfies
    (a) If $A \subseteq X$ then $f(\overline{A}) \subseteq \overline{f(A)}$, or
    (b) If $B \subseteq Y$ then $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

25. Let (X, d) be a metric space and let $a \in X$. Show that
$$\text{if } x, y \in X \text{ then} \quad |d(x, a) - d(y, a)| \leq d(x, y).$$
Conclude that the function $f: X \to \mathbb{R}$ defined by $f(x) = d(x, a)$ is uniformly continuous.

26. Which of the following functions are uniformly continuous?
    (a) $f(x) = \sin x$ on $[0, \infty)$
    (b) $g(x) = \dfrac{1}{1-x}$ on $(0, 1)$
    (c) $h(x) = \sqrt{x}$ on $[0, \infty)$
    (d) $k(x) = \sin(1/x)$, on $(0, 1)$

27. Suppose that A is a dense subset of a metric space (X, d) and $f: A \to \mathbb{R}$ is uniformly continuous. Show that there exists a unique continuous function
$$g: X \to \mathbb{R} \quad \text{such that} \qquad \text{if } x \in A \text{ then } g(x) = f(x).$$

22.10.3 Sequences of functions

1. Let (X, d) and (Y, ρ) be metric spaces. Let $(f_1, f_2, \ldots)$ be a sequence of functions $f_k: X \to Y$ and let $f: X \to Y$ be a function. Show that $(f_1, f_2, \ldots)$ converges uniformly to f
$$\text{if and only if} \qquad \lim_{k\to\infty} (\sup\{\rho(f_k(x), f(x)) \mid x \in X\}) = 0.$$

2. Let (X, d) and (Y, ρ) be metric spaces. Let $(f_1, f_2, \ldots)$ be a sequence of functions $f_k : X \to Y$ and let $f : X \to Y$ be a function. Suppose that $(f_1, f_2, \ldots)$ converges uniformly to $f : X \to Y$. Show that $f : X \to Y$ is continuous.

3. Let $(f_1, f_2, \ldots)$ be a sequence of linear transformations $f_k : \mathbb{R}^n \to \mathbb{R}^m$ which are not identically zero,

$$\text{i.e., if } k \in \mathbb{Z}_{>0} \text{ then there exists } x_k \in \mathbb{R}^n \text{ such that } f_k(x_k) \neq 0.$$

Show that there exists $x \in \mathbb{R}^n$ such that if $k \in \mathbb{Z}_{>0}$ then $f_k(x) \neq 0$.

4. Let $(f_1, f_2, \ldots)$ be a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that

$$\text{if } x \in \mathbb{Q} \text{ then } \{f_1(x), f_2(x), \ldots\} \text{ is unbounded.}$$

Prove that there exists $x \in \mathbb{Q}^c$ such that $\{f_1(x), f_2(x), \ldots\}$ is unbounded.

5. Which of the following sequences of functions converge uniformly on the interval $[0, 1]$

(a) $f_n : [0, 1] \to \mathbb{R}$ given by $f_n(x) = nx^2(1 - x)^n$,
(b) $f_n : [0, 1] \to \mathbb{R}$ given by $f_n(x) = n^2x(1 - x^2)^n$,
(c) $f_n : [0, 1] \to \mathbb{R}$ given by $f_n(x) = n^2x^3e^{-nx^2}$.

6. Determine whether the following sequences of functions converge uniformly.

(a) $f_n : [0, 1] \to \mathbb{R}$ given by $f_n(x) = e^{-nx^2}$, $x \in [0, 1]$;
(b) $g_n : [0, 1] \to \mathbb{R}$ given by $g_n(x) = e^{-x^2/n}$, $x \in [0, 1]$.
(c) $g_n : \mathbb{R} \to \mathbb{R}$ given by $g_n(x) = e^{-x^2/n}$, $x \in \mathbb{R}$.

7. Let (X, d) be a metric space and let $(f_1, f_2, \ldots)$ be a sequence of continuous functions $f_n : X \to \mathbb{R}$.

(a) Give the definition of uniform convergence of the sequence $(f_1, f_2, \ldots)$ to a function $f : X \to \mathbb{R}$.
(b) Prove that if $(f_1, f_2, \ldots)$ converges uniformly to $f : X \to \mathbb{R}$ then f is a continuous function.
(c) Let $f_n : [0, 1] \to \mathbb{R}$ be given by $f_n(x) = \dfrac{1 - x^n}{1 + x^n}$. Find the pointwise limit f of the sequence $(f_1, f_2, \ldots)$.
(d) Let $f_n : [0, 1] \to \mathbb{R}$ be given by $f_n(x) = \dfrac{1 - x^n}{1 + x^n}$. Is the sequence $(f_1, f_2, \ldots)$ uniformly convergent?

8. Let (X, d) be a metric space and let $(f_1, f_2, \ldots)$ be a sequence of continuous functions $f_n : X \to \mathbb{R}$.

(a) Define what it means for the sequence $(f_1, f_2, \ldots)$ to converge uniformly to $f : X \to \mathbb{R}$.
(a) (b)] Suppose that $(f_1, f_2, \ldots)$ is a sequence of continuous functions, $f_n : [0, 1] \to \mathbb{R}$. Assume that $(f_1, f_2, \ldots)$ converges uniformly to $f : [0, 1] \to \mathbb{R}$. Prove that if $x \in [0, 1]$ then

$$\int_0^x f_n(t)dt \quad \text{converges uniformly to} \quad \int_0^x f(t)dt.$$

(c) Let $f_n\colon [0,1] \to \mathbb{R}$ be given by $f_n(x) = \dfrac{x^n}{1+x+x^n}$. Is the sequence $(f_1, f_2, \ldots)$ uniformly convergent?

9. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $\{f_n\}$ be a sequence of functions $f_n : X \to Y$.
   (a) Define what it means for the sequence $(f_1, f_2, \ldots)$ to converge uniformly to a function $f\colon X \to Y$.
   (b) Prove that if each f_n is bounded and $(f_1, f_2, \ldots)$ converges uniformly to f then f is bounded.
   (c) Define $f_n\colon [0,1] \to \mathbb{R}$ by
   $$f_n(x) = \frac{nx^2}{1+nx}.$$
   Find the pointwise limit f of the sequence $(f_1, f_2, \ldots)$ and determine whether the sequence converges uniformly to f.

10. Let $X = C[0,1] = \{f\colon [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$. The supremum metric $d_\infty\colon X \times X \to \mathbb{R}_{\ge 0}$ and the L^1 metric $d_1\colon X \times X \to \mathbb{R}_{\ge 0}$ are defined by
    $$d_\infty(f,g) = \sup\{|f(x) - g(x)| \mid x \in [0,1]\} \qquad \text{and}$$
    $$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx.$$
    Consider the sequence $\{f_1, f_2, f_3, \ldots\}$ in X where
    $$f_n(x) = nx^n(1-x).$$
    (a) Determine whether $(f_1, f_2, \ldots)$ converges in (X, d_1).
    (b) Determine whether $(f_1, f_2, \ldots)$ converges in (X, d_∞).

11. Let (X, d) and (C, ρ) be metric spaces. Let
    $$F = \{\text{functions } f\colon X \to C\} \qquad \text{and define} \quad d_\infty\colon F \times F \to \mathbb{R}_{\ge 0} \cup \{\infty\} \quad \text{by}$$
    $$d_\infty(f,g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$
    (Warning d_∞ is not quite a metric since its target is not $\mathbb{R}_{\ge 0}$.) Let
    $$(f_1, f_2, \ldots) \text{ be a sequence in } F \quad \text{and let} \quad f\colon X \to C$$
    be a function.

    The sequence $(f_1, f_2, \ldots)$ in F *converges pointwise to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

    if $x \in X$ and $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that
    if $n \in \mathbb{Z}_{\ge N}$ then $d(f_n(x), f(x)) < \epsilon$.

    The sequence $(f_1, f_2, \ldots)$ in F *converges uniformly to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

    if $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that
    if $x \in X$ and $n \in \mathbb{Z}_{\ge N}$ then $\rho(f_n(x), f(x)) < \epsilon$.

(a) Show that $(f_1, f_2, \ldots)$ converges pointwise to f if and only if $(f_1, f_2, \ldots)$ satisfies

$$\text{if } x \in X \quad \text{then} \quad \lim_{n\to\infty} \rho(f_n(x), f(x)) = 0.$$

(b) Show that $(f_1, f_2, \ldots)$ converges uniformly to f if and only if $(f_1, f_2, \ldots)$ satisfies

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0.$$

12. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $(f_1, f_2, \ldots)$ be a sequence of functions: $f_n \colon X \to Y$ for $n \in \mathbb{Z}_{>0}$.

    (a) Define what it means for the sequence $(f_1, f_2 \ldots)$ to converge uniformly to a function $f : X \to Y$.
    (b) Define what it means for a function $g \colon X \to Y$ to be bounded.
    (c) Prove that if each f_n is bounded and $(f_1, f_2, \ldots)$ converges uniformly to f, then f is also bounded.
    (d) Define $f_n : [0, 1] \to \mathbb{R}$ for each $n \in \mathbb{Z}_{>0}$ by

    $$f_n(x) = \frac{nx^2}{1 + nx}, \qquad \text{for } x \in [0, 1].$$

    Find the pointwise limit f of the sequence $(f_1, f_2, \ldots)$ and determine whether the sequence converges uniformly to f.

22.10.4 Open dense sets and nowhere dense sets

1. Let (X, d) be a metric space and let $U \subseteq X$ and $V \subseteq X$. Show that if U and V are open and dense then $U \cap V$ is open and dense.

2. Let $X = \mathbb{R}$ with the usual metric and let $U = \mathbb{Q}$ and $V = \mathbb{Q}^c$. Show that U and V are dense and $U \cap B = \emptyset$.

3. Let $X = \mathbb{Q}$ with the usual metric and let $\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}$ be an enumeration of $\mathbb{Q}$. For $n \in \mathbb{Z}_{>0}$ let $Q_n = \mathbb{Q} - \{q_n\}$.

    (a) Show that if $n \in \mathbb{Z}_{>0}$ then Q_n is open and dense.
    (b) Show that $\displaystyle\bigcap_{n\in\mathbb{Z}_{>0}} Q_n = \emptyset$.

4. Let (X, d) be a complete metric space and let $U_1, U_2, U_3, \ldots$ be a sequence of open and dense subsets of X. Show that $\displaystyle\bigcap_{n\in\mathbb{Z}_{>0}} U_n$ is dense in X.

5. Let (X, d) be a complete metric space and let $F_1, F_2, F_3, \ldots$ be a sequence of nowhere dense subsets of X. Show that $\displaystyle\bigcup_{n\in\mathbb{Z}_{>0}} F_n$ has empty interior.

6. Show that $\mathbb{R}$, with the standard topology, cannot be be written as a countable union of nowhere dense sets.

7. Let $X = \mathbb{Q}$, with the standard topology. Let $\mathbb{Q} = \{q_1, q_2, \ldots\}$ be an enumeration of $\mathbb{Q}$. Show that $\{q_n\}$ is nowhere dense. Determine the interior of $\bigcup_{n\in\mathbb{Z}_{>0}} \{q_n\}$.

8. Let (X, d) be a complete metric space and let $(f_1, f_2, f_3, \ldots)$ be a sequence of continuous functions
$$f_n : X \to \mathbb{R}, \qquad \text{for } n \in \mathbb{Z}_{>0}.$$
Assume that if $x \in X$ then $(f_1(x), f_2(x), \ldots)$ is bounded in X. Show that there exists an open set $U \subseteq X$ such that
$$\text{there exists } M \in \mathbb{R}_{>0} \quad \text{such that} \quad \text{if } x \in U \text{ and } n \in \mathbb{Z}_{>0} \text{ then } |f_n(x)| \le M.$$

22.10.5 Connectedness

1. Let X be a set with $Card(X) > 1$.
   (a) Show that X with the discrete topology is disconnected.
   (b) Show that X with the indiscrete topology is connected.

2. Let X_1 and X_2 be the subspaces of $\mathbb{R}$ given by
$$X_1 = \mathbb{R} - \{0\} \qquad \text{and} \qquad X_2 = \mathbb{Q}.$$
Show that X_1 and X_2 are disconnected.

3. Let $Y = \{0, 1\}$ with the discrete topology. Let X be a topological space. Show that X is connected if and only if every continuous function $f : X \to Y$ is constant.

4. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Let $E \subseteq X$. Show that if E is connected then $f(E)$ is connected.

5. Let X be a connected topological space and let $A \subseteq X$. Show that if A is connected then $\overline{A}$, the closure of A, is connected.

6. Let $A = (-\infty, 0)$ and $B = (0, \infty)$ as subsets of $\mathbb{R}$. Show that A is connected, B is connected and $A \cup B$ is not connected.

7. Let X be a topological space. Let $\mathcal{S}$ be a collection of subsets of X such that $\bigcap_{A\in\mathcal{S}} A \neq \emptyset$. Show that $\bigcup_{A\in\mathcal{S}} A$ is connected.

8. Let X be a topological space such that

$$\text{if } x, y \in X \text{ then there exists } A \subseteq X \text{ such that } x \in A,\ y \in A \text{ and } A \text{ is connected.}$$

Show that X is connected.

9. Let X be a topological space. For $x \in X$ let C_x be the connected component containing x.
   (a) Let $y \in X$. Show that C_y is connected and closed.
   (b) Show that the connected components of X partition X.

10. Let X be a set with the discrete topology. Determine (with proof) the connected components of X.

11. Graph each of the following sets and determine (with proof) whether they are connected in $\mathbb{R}^2$?
   (a) $H = \{(x, y) \in \mathbb{R}^2 \mid xy = 1 \text{ and } x, y > 0\}$,
   (b) $L = \{(x, 0) \mid x \in \mathbb{R}\}$,
   (c) $X = H \cup L$,
   (d) $C_n = \big\{(x, y) \in \mathbb{R}^2 \bigm| (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\big\}$, for $n \in \mathbb{Z}$,
   (e) $X = \bigcup_{n \in \mathbb{Z}_{>0}} C_n$.

12. Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ be connected. Show that

$$\text{if } A \subseteq B \subseteq \overline{A} \quad \text{then} \quad B \text{ is connected.}$$

13. Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ and $B \subseteq X$ be connected. Show that

$$\text{if } \overline{A} \cap B \neq \emptyset \quad \text{then} \quad A \cup B \text{ is connected.}$$

14. A point $p \in X$ is called a *cut point* if $X \setminus \{p\}$ is disconnected. Show that the property of having a cut point is a topological property. (A property of a topological space is a *topological property* if it is preserved under homeomorphisms.)

15. Let X be a topological space. Show that if X is path connected then X is connected.

16. Let $X = \{(t, \sin(\pi t)) \mid t \in (0, 2]\} \subseteq \mathbb{R}^2$. Let

$$\varphi\colon \mathbb{R}^2 \to \mathbb{R} \quad \text{be given by} \quad \varphi(x, y) = x.$$

   (a) Show that $\varphi\colon X \to (0, 2]$ is a homeomorphism.
   (b) Show that X is connected.
   (c) Show that $\overline{X}$ is connected.
   (d) Show that $\overline{X}$ is not path connected.

17. Show that the following hold for subsets of a topological space X;
    (a) if subsets A, B are path connected and $A \cap B \neq \emptyset$ then $A \cup B$ is path connected.
    (b) Show that every point of X is contained in a unique path component, which can be defined as the largest path connected subset of X containing this point.
    (c) Give examples to show that the path components need not be open or closed.
    (d) Prove that if X is locally path connected, i.e every point of x is contained in an open set U which is path connected, then every path component is open.
    (e) Conclude that if X is locally path connected, then the path components coincide with the connected components.

18. Prove that if X and Y are path connected then $X \times Y$ is also path connected.

19. A topological space X is defined as *locally connected* if X satsifies:
$$\text{if } x \in X \text{ and } V \subseteq X \text{ is open and } x \in V$$
$$\text{then there exists a connected open set } U \subseteq V \text{ with } x \in U.$$
    (a) Show that if X is locally connected then all the connected components of X are open.
    (b) Assume X is a vector space with a norm. Show that any open subset $A \subseteq X$ is locally connected.

20. Show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic (where $\mathbb{R}$ and $\mathbb{R}^2$ are equipped with the usual topologies).

21. Let A be a countable set. Show that $\mathbb{R}^2 \setminus A$ is path connected.

22. Show that
$$\text{if } A \subseteq \mathbb{R}^n \text{ is open and connected} \quad \text{then} \quad A \text{ is path connected.}$$
[Hint: Fix a point $x_0 \in A$ and consider the set U of all $x \in A$ which can be joined to x_0 by a path in A. Show that U and $A \setminus U$ are open.]

23. A metric space (X, d_X) is *chain connected* if (X, d_X) satisfies
$$\text{if } x, y \in X \text{ and } \varepsilon \in \mathbb{R}_{>0} \text{ then there exists } n \in \mathbb{Z}_{>0} \text{ and } x = x_0,\ x_1,\ x_2, \ldots x_n = y$$
$$\text{such that} \quad \text{if } i \in \{0, 1, \ldots, n-1\} \text{ then } d_X(x_{i+1}, x_i) < \varepsilon.$$
Prove that a compact chain connected metric space is connected.

24. Let
$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \qquad \text{and} \qquad B = \{(x, y) \in \mathbb{R}^2 \mid (x-2)^2 + y^2 < 1\}.$$
Determine whether
$$X = A \cup B, \qquad Y = \overline{A} \cup \overline{B} \qquad \text{and} \qquad Z = \overline{A} \cup B$$
are connected subsets of $\mathbb{R}^2$ with the usual topology.

25. Let X be a connected topological space and let $f: X \to \mathbb{R}$ be a continuous function, where $\mathbb{R}$ has the usual topology. Show that if f takes only rational values then f is a constant function.

26. Show that $X = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$ is not homeomorphic to $\mathbb{R}$ (with the usual topologies). [Hint: consider the effect of removing points from X and $\mathbb{R}$.]

27. Explain why the following pairs of topological spaces are *not* homeomorphic. (Each has the topology induced from the usual embedding into a Euclidean space).

    (a) $\mathbb{R}$ and S^1, where S^1 is the unit circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.
    (b) $(0, \infty)$ and $(0, 1]$.
    (c) $A = \{(0, y) \mid y \in \mathbb{R}\} \cup \{(x, 0) \mid x \in \mathbb{R}\}$ and
    $B = \{(0, y) \mid y \in \mathbb{R}, y \geq 0\} \cup \{(x, 0) \mid x \in \mathbb{R}\}$.

28. Prove that no two of the following spaces are homeomorphic:

    (i) $X = [-1, 1]$ with the topology induced from $\mathbb{R}$;
    (ii) $Y = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ with the topology induced from $\mathbb{R}^2$;
    (iii) $Z = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ with the topology induced from $\mathbb{R}^2$.

29. Define $d: \mathbb{R}_{>1} \to \mathbb{R}_{\geq 0}$ by

$$d(x, y) = \begin{cases} \left| \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{y}} \right|, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

    (a) Show that d is a metric.
    (b) Show that $\phi: (1, \infty) \to (0, 1)$ defined by $\phi(x) = \dfrac{1}{\sqrt{x}}$ is an isometry.
    (c) Determine (with proof) if the metric space $((1, \infty), d)$ is connected.
    (d) Determine (with proof) if the metric space $((1, \infty), d)$ is compact.

30. (a) Let X be a topological space and let A and B be connected subsets of X such that $A \cap B \neq \emptyset$. Prove that $A \cup B$ is a connected subset of X.
    (b) Let $f: X \to Y$ be a continuous map between topological spaces. Prove that if X is compact then $f(X)$ is compact.

31. Show that $\mathbb{Q}$, with the standard topology, is totally disconnected (i.e. each connected component contains only one point).

32. Show that a subset of $\mathbb{R}$ is connected if and only if it is an interval.

33. Carefully state the Intermediate Value Theorem.

34. State and prove the Intermediate Value Theorem.

35. Let X be a connected topological space and let $f\colon X \to \mathbb{R}$ be a continuous function. Show that if $x, y \in X$ and $r \in \mathbb{R}$ such that $f(x) \le r \le f(y)$ then there exists $c \in X$ such that $f(c) = r$.

36. (a) Let (X, d_X) and (Y, d_Y) be a metric spaces and let $f\colon X \to Y$ be a function. Let $E \subseteq X$. Prove that if $f\colon X \to Y$ is continuous and E is connected then $f(E)$ is connected.
    (b) Carefully state the intermediate value theorem.
    (c) Prove the intermediate value theorem.

22.10.6 Hausdorff and normal spaces

1. Let (X, d) be a metric space.
    (a) Define the metric space topology $\mathcal{T}$ on X.
    (b) Define Hausdorff and show that the topological space $(X, \mathcal{T})$ is Hausdorff.
    (c) Define normal and show that the topological space $(X, \mathcal{T})$ is normal.
    (d) Define first countable and show that the topological space $(X, \mathcal{T})$ is first countable.
    (e) Give an example (with proof) of a topological space $(Y, \mathcal{U})$ which is not Hausdorff.
    (f) Give an example (with proof) of a topological space $(Y, \mathcal{U})$ which is not normal.
    (g) Give an example (with proof) of a topological space $(Y, \mathcal{U})$ which is not first countable.

2. (a) Define topological space and Hausdorff topological space.
    (b) Give an example of a topological space which is not Hausdorff.
    (c) Show that metric spaces are Hausdorff (with the metric space topology).

22.10.7 Distances and diameters

1. Let A be a nonempty subset of a metric space (X, d). Show that
    (a) $x \in \overline{A}$ if and only if $d(x, A) = 0$.
    (b) Show that $\mathrm{diam}(A) = \mathrm{diam}(\overline{A})$.

2. Show that if $A \subseteq X$ then $\mathrm{diam}(A) = \mathrm{diam}(\overline{A})$. Does $\mathrm{diam}(A) = \mathrm{diam}(A^\circ)$?

3. Let (X, d) be a metric space and let A be a non-empty subset of X. Recall that for each $x \in X$, the distance from x to A is
$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$
    (a) Prove that $\overline{A} = \{x \in X : d(x, A) = 0\}$.
    (b) Prove that $|d(x, A) - d(y, A)| \le d(x, y)$ for all $x, y \in X$. (Hint: first show that $d(x, A) \le d(x, y) + d(y, A)$.)
    (c) Deduce the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is continuous.
    (d) Show that if $x \notin \overline{A}$ then $U = \{y \in X \mid d(y, A) < d(x, A)\}$ is an open set in X such that $\overline{A} \subseteq U$ and $x \notin U$.

4. Let (X, d) be a metric space and fix a point $p \in X$.
   (a) Prove that the function $f : X \to \mathbb{R}$ defined by $f(x) = d(p, x)$ is continuous, where $\mathbb{R}$ has the usual metric.
   (b) Let A be a non-empty compact subset of X.
      (i) Prove that there exists a point $a \in A$ such that
      $$d(p, a) = \inf\{d(p, x) \mid x \in A\}.$$
      (ii) Give an example to show that the point a as in (i) need not be unique.