10 New spaces from old spaces

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$.

10.1 Subspaces

10.1.1 Topological spaces

Let $(X, \mathcal{T})$ be a topological space and let Y be a subset of X. The *subspace topology on* Y is

$$\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}.$$

10.1.2 Uniform spaces

Let $(X, \mathcal{X})$ be a uniform space and let Y be a subset of X. The *subspace uniformity on* Y is

$$\mathcal{X}_Y = \{V \cap (Y \times Y) \mid V \in \mathcal{X}\}.$$

10.2 Vector spaces

Let X be a $\mathbb{K}$-vector space. A *subspace*, or *linear subspace*, of X is a subset $V \subseteq X$ such that

(a) If $v_1, v_2 \in V$ then $v_1 + v_2 \in V$,

(b) If $v \in V$ and $c \in \mathbb{K}$ then $cv \in V$.

A better term might be "vector subspace" to avoid confusion with subspace topologies and subspace uniformities, which are much more general than vector spaces, but the standard language is to use the term "subspace" for "linear subspace".

10.3 Product spaces

10.3.1 Vector spaces

Let X and Y be $\mathbb{K}$-vector spaces. The *direct sum* of X and Y is the $\mathbb{K}$-vector space $X \oplus Y$ given by the set $X \times Y$ with addition and scalar multiplication given by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \qquad \text{and} \qquad c(x, y) = (cx, cy),$$

for $x, x_1, x_2 \in X$, $y, y_1, y_2 \in Y$ and $c \in \mathbb{K}$.

10.3.2 Topological spaces

Let $(X, \mathcal{T})$ and $(Y, \mathcal{Q})$ be topological spaces and let $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$. The *product topology on* $X \times Y$ is

$$\mathcal{T} = \{\text{unions of sets in } \mathcal{B}\}, \qquad \text{where} \quad \mathcal{B} = \{U \times V \mid U \in \mathcal{T}, V \in \mathcal{Q}\}.$$

10.3.3 Uniform spaces

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be uniform spaces. Let $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ and

$$\begin{array}{rccc} \gamma_X\colon & (X\times Y)\times(X\times Y) & \to & X\times X \\ & ((x_1,y_1),(x_2,y_2)) & \mapsto & (x_1,x_2) \end{array} \quad \text{and} \quad \begin{array}{rccc} \gamma_Y\colon & (X\times Y)\times(X\times Y) & \to & Y\times Y \\ & ((x_1,y_1),(x_2,y_2)) & \mapsto & (y_1,y_2). \end{array}$$

The *product uniformity on $X \times Y$* is

$$\mathcal{Z} = \{\text{subsets of } (X\times Y)\times(X\times Y) \text{ which contain a set of } \mathcal{B}\}, \qquad \text{where}$$

$$\mathcal{B} = \{\gamma_X^{-1}(D_1)\cap\cdots\cap\gamma_X^{-1}(D_\ell)\cap\gamma_Y^{-1}(E_1)\cap\cdots\cap\gamma_Y^{-1}(E_m) \mid \ell, m \in \mathbb{Z}_{>0}, D_i \in \mathcal{X}, E_j \in \mathcal{Y}\}.$$

10.4 Bounded linear operators $B(V, W)$

Let V and W be $\mathbb{K}$-vector spaces. A *linear transformation from V to W* is a function $T\colon V \to W$ such that

(a) If $v_1, v_2 \in V$ then $T(v_1 + v_2) = T(v_1) + T(v_2)$,

(b) If $v \in V$ and $c \in \mathbb{K}$ then $T(cv) = cT(v)$.

Let V and W be normed vector spaces. The space of *bounded operators from V to W* is the normed vector space

$$B(V,W) = \{\text{linear transformations } T\colon V \to W \mid \|T\| < \infty\} \qquad \text{where}$$

$$\|T\| = \sup\left\{\frac{\|Tv\|}{\|v\|} \mid v \in V \text{ and } v \neq 0\right\}.$$

Theorem 10.1. *Let $(V, \|\ \|)$ and $(W, \|\ \|)$ be normed vector spaces and let*

$$B(V,W) = \{\textit{linear transformations } T\colon V \to W \mid \|T\| < \infty\} \qquad \textit{where}$$

$$\|T\| = \sup\left\{\frac{\|Tv\|}{\|v\|} \mid v \in V \textit{ and } v \neq 0\right\}.$$

If W is complete then $B(V, W)$ is complete.

10.4.1 Dual spaces V^*

Let V be a normed $\mathbb{R}$-vector space. The space of *bounded linear functionals on V*, or *the dual of V*, is the normed vector space

$$V^* = B(V,\mathbb{R}) = \{\text{linear transformations } T\colon V \to \mathbb{R} \mid \|T\| < \infty\} \qquad \text{where}$$

$$\|T\| = \sup\left\{\frac{\|Tv\|}{\|v\|} \mid v \in V \text{ and } v \neq 0\right\}.$$

10.5 Function spaces

Let (X, d_X) and (Y, d_Y) be metric spaces and let

$$\mathcal{BC}(X,Y) = \{f\colon X \to Y \mid f \text{ is continuous and } f(X) \text{ is bounded in } Y\},$$

with metric $d_\infty\colon \mathcal{BC}(X,Y) \times \mathcal{BC}(X,Y) \to \mathbb{R}_{\geq 0}$ given by

$$d_\infty(f,g) = \sup\{d_Y(f(x), g(x)) \mid x \in X\}.$$

10.6 Notes and references

The subspace topology, the product topology, and the minimal topology containing a collection of sets are discussed in [Bou, Top. Ch. I §2 no. 3 Examples]. See also [Bou, Top. Ch. I §3 no. 1] for subspaces and [Bou, Top. Ch. I §4 no. 1] for products in the category of topological spaces. The subspace uniformity and the product uniformity are defined in [Bou, Top. Ch. II §2 no. 4 Def. 3] and [Bou, Top. Ch. II §2 no. 6 Def. 4].

In practice, it is often more convenient to work with a good set of *generators* of a topology rather than with *all* the sets in a topology. Let $(X, \mathcal{T})$ be a topological space. A *union generating set for* $\mathcal{T}$, or a *basis of* $\mathcal{T}$, is a collection $\mathcal{B}$ of subsets of X such that

$$\mathcal{T} = \{\text{unions of sets in } \mathcal{B}\}.$$

The collection of *rectangles*

$$\mathcal{B} = \{U \times V \mid U \in \mathcal{T} \text{ and } V \in \mathcal{T}\}$$

is an union generating set for the product topology on $X \times Y$.

In practice, it is often more convenient to work with a good set of *generators* of the neighborhood filter rather than with *all* the sets in a neighborhood filter. Let $(X, \mathcal{T}_X)$ be a topological spaces, let $x \in X$ and let $\mathcal{N}(x)$ denote the neighborhood filter of x. An *inclusion generating set for* $\mathcal{N}(x)$ is a collection $\mathcal{B}(x)$ of subsets of X such that

$$\mathcal{N}(x) = \{\text{subsets of } X \text{ that contain a set in } \mathcal{B}(x)\}.$$

If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces then

$$\mathcal{B}(x, y) = \{N \times P \mid N \in \mathcal{N}(x) \text{ and } P \in \mathcal{N}(y)\}$$

is an inclusion generating set for $\mathcal{N}((x, y))$ in the product topology on $X \times Y$.