Orthogonals and projections

Let $(H, \langle\,\rangle)$ be a Hilbert space.
Let $\overline{W}$ be a closed subspace of H.

A *projection onto* $\overline{W}$ is a function $P: H \to H$ such that

(P1) If $x \in H$ then $P(x) \in \overline{W}$

(P2) If $x \in H$ and $w \in \overline{W}$ then $\langle x, w\rangle = \langle P(x), w\rangle$.

Let $(e_1, e_2, e_3, \ldots)$ be an orthonormal sequence in H and let

$$\overline{W} = \overline{\mathrm{span}\{e_1, e_2, \ldots\}} \quad \text{and}$$

$$\overline{W}^\perp = \{h \in H \mid \text{if } w \in \overline{W} \text{ then } \langle h, w\rangle = 0\}.$$

Theorem Define $P: H \to H$ by

$$P(x) = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n$$

then P is a projection onto $\overline{W} = \overline{\mathrm{span}\{e_1, e_2, \ldots\}}$.

Proof (a) To show: $P: H \to H$ is a function

(b) To show: If $x \in H$ then $P(x) \in \overline{W}$

(c) To show: If $x \in H$ and $w \in \overline{W}$ then

$$\langle P(x), w\rangle = \langle x, w\rangle.$$

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<u>Bessel's inequality</u> If $x \in H$ then

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \le \|x\|^2$$

<u>Proof</u> Assume $x \in H$.

To show: $\lim_{k\to\infty} \left(\sum_{n=1}^{k} |\langle x, e_n\rangle|^2\right) \le \|x\|^2$.

For $k \in \mathbb{Z}_{>0}$ let

$$x_k = \sum_{n=1}^{k} \langle x, e_n\rangle e_n \quad \text{so that}$$

$$\|x_k\|^2 = \sum_{n=1}^{k} \langle x, e_n\rangle \overline{\langle x, e_n\rangle} = \sum_{n=1}^{k} |\langle x, e_n\rangle|^2.$$

To show: $\|x_k\|^2 \le \|x\|^2$.

$$\langle x - x_k, x_k\rangle = \langle x, x_k\rangle - \langle x_k, x_k\rangle$$
$$= \sum_{n=1}^{k} \langle x, e_n\rangle \overline{\langle x, e_n\rangle} - \sum_{n=1}^{k} \langle x, e_n\rangle \overline{\langle x, e_n\rangle} = 0$$

and

$$\|x\|^2 = \langle x, x\rangle = \langle x_k + (x - x_k), x_k + (x - x_k)\rangle$$
$$= \langle x_k, x_k\rangle + \langle x_k, x - x_k\rangle + \langle x - x_k, x_k\rangle + \langle (x - x_k), (x - x_k)\rangle$$
$$= \|x_k\|^2 + 0 + 0 + \|x - x_k\|^2 \ge \|x_k\|^2.$$

So, ~~if~~ $k \in \mathbb{Z}_{>0}$ then $\|x_k\|^2 \le \|x\|^2$.

So $\lim_{n\to\infty} \left(\sum_{n=1}^{k} |\langle x, e_n\rangle|^2\right) \le \|x\|^2$.

So $\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \le \|x\|^2$.

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(a) To show: If $x \in H$ then $P(x) = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ exists in H.

Assume $x \in H$. Let
$$x_k = \sum_{n=1}^{k} \langle x, e_n \rangle e_n, \quad \text{for } k \in \mathbb{Z}_{>0}.$$

To show: $\lim_{k \to \infty} x_k$ exists in H.

Using that H is complete, (

To show: $(x_1, x_2, \ldots)$ is a Cauchy sequence

By Bessel's inequality,

$(\|x_1\|, \|x_2\|, \|x_3\|, \ldots)$ is an increasing sequence in $\mathbb{R}_{\geq 0}$ bounded by $\|x\|$.

So $(\|x_1\|, \|x_2\|, \ldots)$ converges in $\mathbb{R}_{\geq 0}$. Let
$$\ell = \lim_{k \to \infty} \|x_k\|$$

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{>N}$ then $\|x_r - x_s\| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq N}$ then $\|x_r - x_s\| < \varepsilon$.

Let $N \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\geq N}$ then
$$|\ell^2 - \|x_k\|^2| < \frac{\varepsilon}{10}$$

To show: If $r,s \in \mathbb{Z}_{\geq N}$ then $\|x_r - x_s\| < \varepsilon$.

Assume $r,s \in \mathbb{Z}_{\geq N}$.

To show: $\|x_r - x_s\| < \varepsilon$.

$$\|x_r - x_s\|^2 = \Big\| \sum_{j=1}^{r} \langle x, e_j\rangle e_j - \sum_{j=1}^{s} \langle x, e_j\rangle e_j \Big\|^2$$

$$= \Big\| \sum_{j=r+1}^{s} \langle x, e_j\rangle e_j \Big\|^2 = \sum_{j=r+1}^{s} |\langle x, e_j\rangle|^2$$

$$= \big| \|x_r\|^2 - \|x_s\|^2 \big| = \big| \|x_s\|^2 - l^2 + l^2 - \|x_r\|^2 \big|$$

$$\leq \big| \|x_s\|^2 - l^2 \big| + \big| l^2 - \|x_r\|^2 \big| < \frac{\varepsilon}{10} + \frac{\varepsilon}{10} < \varepsilon.$$

So $(x_1, x_2, \ldots)$ is a Cauchy sequence in H.

So $\lim_{k\to\infty} x_k$ exists in H.

So $P(x) = \sum_{j=1}^{\infty} \langle x, e_n\rangle e_n$ exists in H.

So $P: H \to H$ is a function.

(b) To show: If $x \in H$ then $P(x) \in \overline{W}$.

Assume $x \in H$.

Since
$$x_k = \sum_{j=1}^{k} \langle x, e_j\rangle e_j \in \mathrm{span}\{e_1, e_2, \ldots\} \subseteq \overline{W}$$

then $P(x) = \lim_{k\to\infty} x_k \in \overline{W}$.

(c) To show: If $x \in H$ and $w \in \overline{W}$ then

$$\langle x, w\rangle = \langle P(x), w\rangle.$$

Assume $x \in H$ and $w \in \overline{W}$.

~~Let $(b_1, b_2, \ldots)$ be~~

To show: $\langle x - P(x), w\rangle = 0$.

$$\langle x - P(x), e_r\rangle = \langle x, e_r\rangle - \langle \lim_{k\to\infty} x_k, e_r\rangle$$

$$= \langle x, e_r\rangle - \lim_{k\to\infty} \langle x_k, e_r\rangle \quad \left(\begin{array}{l}\text{since } \langle\ ,\ \rangle \\ \text{is continuous}\end{array}\right)$$

$$= \langle x, e_r\rangle - \lim_{k\to\infty} \langle x, e_r\rangle$$

$$= \langle x, e_r\rangle - \langle x, e_r\rangle = 0.$$

Let $(w_1, w_2, \ldots)$ be a sequence in W such that

$$w = \lim_{n\to\infty} w_n.$$

~~Then~~ If $w_n = \sum c_j e_j$ then

$$\langle x - P(x), w_n\rangle = \sum_{j=1}^{\ell} \bar{c}_j \langle x - P(x), e_j\rangle = 0.$$

So

$$\langle x - P(x), w\rangle = \lim_{n\to\infty} \langle x - P(x), w_n\rangle = \lim_{n\to\infty} 0 = 0.$$

So $\langle x, w\rangle = \langle P(x), w\rangle$.

So $P: H \to H$ is a projection.