Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Hilbert space duals 08.08.2022 MHS Lect 7 ① A. Ram

Proposition Let $(H, \langle,\rangle)$ be a Hilbert space. Then

$$\begin{aligned} \Phi: H &\longrightarrow H^* \\ x &\longmapsto \varphi_x \end{aligned} \qquad \text{where } \begin{aligned} \varphi_x: H &\longrightarrow \mathbb{K} \\ h &\longmapsto \langle h, x\rangle. \end{aligned}$$

is a skew-linear bijective isometry. and $\|\Phi\|=1$.

Proof To show: (a) Φ is skew linear
(b) Φ is an isometry
(c) $\|\Phi\|=1$
(d) Φ is injective
(e) Φ is surjective

(a) To show: (aa) If $a, b \in H$ then $\varphi_{a+b} = \varphi_a + \varphi_b$.
(ab) If $a \in H$ and $c \in \mathbb{K}$ then $\varphi_{ca} = \bar{c}\varphi_a$

(aa) Assume $a, b \in H$

To show: ~~If $a, b \in H$ then~~ $\varphi_{a+b} = \varphi_a + \varphi_b$.

To show: If $h \in H$ then $\varphi_{a+b}(h) = (\varphi_a + \varphi_b)(h)$.

Assume $h \in H$

To show: $\varphi_{a+b}(h) = (\varphi_a + \varphi_b)(h)$

$$\varphi_{a+b}(h) = \langle h, a+b\rangle = \langle h, a\rangle + \langle h, b\rangle$$
$$= \varphi_a(h) + \varphi_b(h) = (\varphi_a + \varphi_b)(h).$$

(ab) To show: If $a \in H$ and $c \in \mathbb{K}$ then $\varphi_{ca} = \bar{c}\varphi_a$.

Assume $a \in H$ and $c \in \mathbb{K}$.

To show: $\varphi_{ca} = \bar{c}\varphi_a$.

To show: If $h \in H$ then $\varphi_{ca}(h) = \bar{c}\varphi_a(h)$

Assume $h \in H$.

To show: $\varphi_{ca}(h) = \bar{c}\varphi_a(h)$.

$$\varphi_{ca}(h) = \langle h, ca \rangle = \bar{c}\langle h, a \rangle = \bar{c}\varphi_a(h).$$

(b) To show: Φ is an isometry.

To show: If $x \in H$ then $\|x\| = \|\varphi_x\|$.

Assume $x \in H$.

To show: $\|\varphi_x\| = \|x\|$.

To show: (ba) $\|\varphi_x\| \le \|x\|$

(bb) $\|\varphi_x\| \ge \|x\|$.

(ba) If $h \in H$ then, by Cauchy-Schwarz

$$|\varphi_x(h)| = |\langle h, x \rangle| \le \|h\| \cdot \|x\|.$$

So

$$\frac{|\varphi_x(h)|}{\|h\|} \le \|x\|.$$

So

$$\sup\left\{ \frac{|\varphi_x(h)|}{\|h\|} \ \Big|\ h \in H \right\} \le \|x\|.$$

So $\|\varphi_x\| \le \|x\|$.

(b) Since
$$|\varphi_x(x)| = |\langle x, x\rangle| = \|x\|^2 = \|x\| \cdot \|x\|$$
then $\dfrac{|\varphi_x(x)|}{\|x\|} = \|x\|$.

So $\sup\left\{ \dfrac{|\varphi_x(h)|}{\|h\|} \;\middle|\; h \in H \right\} \ge \|x\|$.

So $\|\varphi_x\| \ge \|x\|$.

So $\|\varphi_x\| = \|x\|$.

So $\|\Phi(x)\| = \|x\|$.

So Φ is an isometry.

(c) To show: $\|\Phi\| = 1$.

Using that $\|\varphi_x\| = \|x\|$ from part (b),
$$\|\Phi\| = \sup\left\{ \frac{\|\Phi(x)\|_{H^*}}{\|x\|_H} \;\middle|\; x \in H \right\}$$
$$= \sup\left\{ \frac{\|\varphi_x\|_{H^*}}{\|x\|_H} \;\middle|\; x \in H \right\} = \sup\left\{ \frac{\|x\|_H}{\|x\|_H} \;\middle|\; x \in H \right\}$$
$$= \sup\{ 1 \mid x \in H \} = 1.$$

So $\|\Phi\| = 1$.

(d) To show: Φ is injective.

To show: If $a, b \in H$ and $\varphi_a = \varphi_b$ then $a = b$.

Assume $a, b \in H$ and $\varphi_a = \varphi_b$.

To show: $a = b$.

To show: $\|a - b\| = 0$.

$$\|a-b\| = \|\varphi_{a-b}\| = \|\varphi_a - \varphi_b\| = \|0\| = 0.$$

So $a = b$.

So Φ is injective

(e) To show: Φ is surjective.

To show: If $\varphi \in H^*$ then there exists $a \in H$ such that $\varphi = \varphi_a = \Phi(a)$.

Assume $\varphi \in H^*$.

To show: There exists $a \in H$ such that $\varphi = \Phi(a)$.

Case 1: $\varphi = 0$.

Let $a = 0$.

Then $\varphi = 0 = \varphi_0 = \Phi(0) = \Phi(a)$

Case 2: $\varphi \neq 0$

Since φ is bounded then φ is continuous.

08.08.2022
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(5)

Since $\{0\}$ is closed in $\mathbb{K}$ then

$$\ker\varphi = \varphi^{-1}(\{0\}) \text{ is closed in } H.$$

By the orthogonal decomposition theorem

$$H = \ker\varphi \oplus (\ker\varphi)^\perp$$

Let

$$b \in (\ker\varphi)^\perp \text{ with } b \neq 0$$

and let $\quad a = \dfrac{\varphi(b)}{\|b\|^2} b.$

To show: $\varphi = \varphi_a$.

To show: If $h \in H$ then $\varphi(h) = \varphi_a(h)$.

Assume $h \in H$.

Then

$$h = \left(h - \frac{\varphi(h)}{\varphi(a)} a\right) + \frac{\varphi(h)}{\varphi(a)} a, \quad \text{with}$$

$$h - \frac{\varphi(h)}{\varphi(a)} a \in \ker\varphi \quad \text{and} \quad \frac{\varphi(h)}{\varphi(a)} a \in (\ker\varphi)^\perp$$

To show: $\varphi(h) = \varphi_a(h)$

$$\varphi_a(h) = \langle h, a\rangle = \left\langle \left(h - \frac{\varphi(h)}{\varphi(a)} a\right) + \frac{\varphi(h)}{\varphi(a)} a, a \right\rangle$$

$$= \left\langle h - \frac{\varphi(h)}{\varphi(a)} a, a \right\rangle + \left\langle \frac{\varphi(h)}{\varphi(a)} a, a \right\rangle$$

$$= 0 + \left\langle \frac{\varphi(h)}{\varphi(a)} a, a \right\rangle, \text{ since } h - \frac{\varphi(h)}{\varphi(a)} a \in \ker\varphi \text{ and } a \in (\ker\varphi)^\perp$$

$$= \frac{\varphi(h)}{\varphi(a)}\langle a,a\rangle = \frac{\varphi(h)}{\varphi(a)}\left\langle \frac{\overline{\varphi(b)}}{\|b\|^2}b, \frac{\overline{\varphi(b)}}{\|b\|^2}b\right\rangle$$

$$= \frac{\varphi(h)}{\varphi(a)}\frac{\overline{\varphi(b)}}{\|b\|^2}\frac{\varphi(b)}{\|b\|^2}\langle b,b\rangle$$

$$= \frac{\varphi(h)}{\varphi(a)}\frac{\overline{\varphi(b)}}{\|b\|^2}\varphi(b)\frac{\|b\|^2}{\|b\|^2} = \frac{\varphi(h)}{\varphi(a)}\frac{\overline{\varphi(b)}\varphi(b)}{\|b\|^2}$$

$$= \frac{\varphi(h)}{\varphi(a)}\varphi\left(\frac{\overline{\varphi(b)}}{\|b\|^2}b\right) = \frac{\varphi(h)}{\varphi(a)}\varphi(a) = \varphi(h).$$

So $\varphi_a = \varphi$.

So $\varphi = \Phi(a)$.

So Φ is surjective. //

Theorems that appeared along the way.

<u>Theorem</u> Let H be a Hilbert space.
Let W be a subset of H

(a) $W^\perp$ is a closed subspace of H.

(b) W is a closed subspace of H if and only if

$$H = W \oplus W^\perp.$$

<u>Theorem</u> Let $(H, \langle \ \rangle)$ be a $\mathbb{K}$-vector space with a positive definite Hermitian form.

(a) If $x, y \in H$ and $\langle x, y \rangle = 0$ then
$$\|x\|^2 + \|y\|^2 = \|x+y\|^2$$

(b) If $x, y \in H$ then
$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

(c) If $x, y \in H$ then
$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

(d) If $x, y \in H$ then $\|x+y\| \leq \|x\| + \|y\|$.