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MHS Lect 6
A. Ram

Completions: Theorem

Let (X, d) be a metric space.

Let $(\hat{X}, \hat{d}, \iota)$ be the metric space

$$\hat{X} = \{\text{Cauchy sequences } \vec{x} \text{ in } X\}$$

with the function $\iota: X \to \hat{X}$ given by

$$\iota(x) = (x, x, x, \ldots), \quad \text{for } x \in X,$$

where $\hat{X}$ has the metric $\hat{d}: \hat{X} \times \hat{X} \to \mathbb{R}_{\geq 0}$ given by

$$\hat{d}(\vec{x}, \vec{y}) = \lim_{n \to \infty} d(x_n, y_n)$$

and Cauchy sequences

$$\vec{x} = (x_1, x_2, \ldots) \text{ and } \vec{z} = (z_1, z_2, \ldots)$$

are equal in $\hat{X}$,

$$\vec{x} = \vec{z} \quad \text{if} \quad \lim_{n \to \infty} d(x_n, z_n) = 0.$$

Then
(a) $(\hat{X}, \hat{d})$ is a complete metric space
(b) $\overline{\iota(X)} = \hat{X}$.

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<u>Proof</u> To show:

(a) $(\hat{X}, \hat{d})$ is a metric space.

(b) $(\hat{X}, \hat{d})$ is complete.

(c) $\iota : X \to \hat{X}$ is an isometry.

(d) $\overline{\iota(X)} = \hat{X}$.

(c) To show: $\iota : X \to \hat{X}$ is an isometry

To show: If $x, y \in X$ then $\hat{d}(\iota(x), \iota(y)) = d(x,y)$.

Assume $x, y \in X$.

To show: $\hat{d}(\iota(x), \iota(y)) = d(x,y)$.

$$\hat{d}(\iota(x), \iota(y)) = \hat{d}((x,x,\ldots), (y,y,y,\ldots))$$

$$= \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x,y) = d(x,y).$$

So ι is an isometry (of metric spaces).

(d) To show: $\overline{\iota(X)} = \hat{X}$.

To show: (da) $\overline{\iota(X)} \subseteq \hat{X}$

(db) $\hat{X} \subseteq \overline{\iota(X)}$

(da) Since $\iota: X \to \hat{X}$ is a function then $\iota(X) \subseteq \hat{X}$.

By the definition of closure in $\hat{X}$,

$\overline{\iota(X)} \subseteq \hat{X}$.

(db) To show: If $\vec{x} \in \hat{X}$ then $\vec{x} \in \overline{\iota(X)}$.

Assume $x \in \hat{X}$.

To show: $\vec{x} \in \overline{\iota(X)}$.

To show: There exists a sequence

$\vec{y}_1, \vec{y}_2, \ldots \in$ ~~[illegible]~~ $\iota(X)$ such that $\lim_{n\to\infty} \vec{y}_n = \vec{x}$.

To show: There exists a sequence

$\iota(z_1), \iota(z_2), \ldots$ in $\iota(X)$ such that $\lim_{n\to\infty} \iota(z_n) = \vec{x}$

Let $\vec{x} = (x_1, x_2, x_3, \ldots)$ and let

$z_1 = x_1$
$z_2 = x_2$
$z_3 = x_3$
$\vdots$

so that

$\vec{y}_1 = \iota(z_1) = (z_1, z_1, z_1, \ldots)$
$\vec{y}_2 = \iota(z_2) = (z_2, z_2, z_2, \ldots)$
$\vec{y}_3 = \iota(z_3) = (z_3, z_3, z_3, \ldots)$
$\vdots$

To show: $\lim_{n\to\infty} \iota(z_n) = \vec{x}$.

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To show: $\lim_{n\to\infty} \hat{d}(z(z_n), \vec{x}) = 0$.

To show: If $\varepsilon \in \mathbb{E}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $\hat{d}(z(z_n), \vec{x}) = 0$.

Assume $\varepsilon \in \mathbb{E}$.

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $\hat{d}(z(z_n), \vec{x}) = 0$.

Using that $\vec{x} \in \hat{X}$, then $\vec{x} = (x_1, x_2, x_3, \ldots)$ is a Cauchy sequence in X and so there is an $M \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{>N}$ then $d(x_r, x_s) < \frac{\varepsilon}{2}$.

Let $N = M$.

To show: If $n \in \mathbb{Z}_{\geq N}$ then $\hat{d}(z(z_n), \vec{x}) < \varepsilon$.

Assume $n \in \mathbb{Z}_{\geq N}$

To show: $\hat{d}(z(z_n), \vec{x}) < \varepsilon$.

To show: $\lim_{k\to\infty} d(z_n, x_k) < \varepsilon$.

$$\lim_{k\to\infty} d(z_n, x_k) = \lim_{n\to\infty} d(x_n, x_k) \leq \frac{\varepsilon}{2} < \varepsilon,$$

since $d(x_n, x_k) < \frac{\varepsilon}{2}$ for $k > N$.

So $\lim_{n\to\infty} \hat{d}(\iota(z_n), \vec{x}) = 0$.

So $\lim_{n\to\infty} \iota(z_n) = \vec{x}$ and $\lim_{n\to\infty} \vec{y}_n = \vec{x}$.

So $\vec{x} \in \overline{\iota(X)}$.

Hence $\hat{X} \subseteq \overline{\iota(X)}$.

So $\overline{\iota(X)} = \hat{X}$.