## Duals

Use the notation

$$\varphi_x(y) = \langle x,y\rangle = \sum_{i=1}^{\infty} x_i y_i$$

for $x=(x_1,x_2,\dots)$ and $y=(y_1,y_2,\dots)$ in $\mathbb{R}^\infty$.

Let $p,q \in \mathbb{R}_{>0}$ with $\frac{1}{p}+\frac{1}{q}=1$. The Hölder-Minkowski inequality says that

if $x\in \ell^p$ and $y \in \ell^q$ then $|\langle x,y\rangle| \le \|x\|_p \cdot \|y\|_q$.

If $\varphi \in (\ell^q)^*$ and $y\in \ell^q$ then

$$|\varphi(y)| = \left|\varphi\left(\sum_{i=1}^{\infty} y_i e_i\right)\right| = \left|\sum_{i=1}^{\infty} y_i \varphi(e_i)\right|$$

$$= |\langle y, (\varphi(e_1), \varphi(e_2),\dots)\rangle|$$

$$\le \|y\|_q \cdot \|(\varphi(e_1),\varphi(e_2),\dots)\|_p .$$

Thus

$$\|\varphi\| \le \|(\varphi(e_1),\varphi(e_2),\dots)\|_p .$$

Let

$s=(s_1,s_2,\dots)$, where $s_i = \mathrm{sgn}(\varphi(e_i))\,|\varphi(e_i)|^{p/q}$.

Then

$$|\varphi(s)| = \Big|\varphi\Big(\sum_{i=1}^{\infty} s_i e_i\Big)\Big| = \Big|\sum_{i=1}^{\infty} s_i \varphi(e_i)\Big|$$

$$= \Big|\sum_{i=1}^{\infty} |\varphi(e_i)|\,|\varphi(e_i)|^{p/q}\Big|$$

$$= \sum_{i=1}^{\infty} |\varphi(e_i)|\,|\varphi(e_i)|^{p(1-\frac{1}{p})}$$

$$= \Big(\sum_{i=1}^{\infty} |\varphi(e_i)|^{p}\Big)^{\frac{1}{p}+\frac{1}{q}}$$

$$= \|(\varphi(e_1),\varphi(e_2),\dots)\|_p \Big(\sum_{i=1}^{\infty} \big(|\varphi(e_i)|^{p/q}\big)^{q}\Big)^{\frac{1}{q}}$$

$$= \|(\varphi(e_1),\varphi(e_2),\dots)\|_p \cdot \|s\|_q,$$

which gives that

$$\|\varphi\| \geq \|(\varphi(e_1),\varphi(e_2),\dots)\|.$$

So

$$\|\varphi\| = \|(\varphi(e_1),\varphi(e_2),\dots)\|.$$

If $x \in l^p$ then

$$\|\varphi_x\| = \|x\|_p,$$

since $x = (x_1, x_2, \dots) = (\varphi_x(e_1), \varphi_x(e_2), \dots)$.

These computations show that there are well defined maps (isometries of normed vector spaces)

$$\Phi_p : \ell^p \to (\ell^q)^* \quad \text{and} \quad \Psi_p : (\ell^q)^* \to \ell^p$$
$$x \longmapsto \varphi_x \qquad\qquad \varphi \mapsto (\varphi(e_1), \varphi(e_2), \dots)$$

If $\varphi \in (\ell^q)^*$ then $\varphi$ is determined by the values $x_i = \varphi(e_i)$ because $\varphi$ is continuous (bounded) and $\overline{\tau_i} = \ell^q$. Thus $\varphi = \varphi_x = \Phi_p(x)$ and $\Phi_p$ is surjective.

Since $(\varphi_x(e_1), \varphi_x(e_2), \dots) = (x_1, x_2, \dots) = x$ then the functions $\Psi_p$ and $\Phi_p$ are inverse functions.

Since $\Phi_p$ and $\Psi_p$ are isometries then

$$\|\Phi_p\| = 1 \quad \text{and} \quad \|\Psi_p\| = 1.$$