Union generating sets for topologies

Proposition: Let X be a set. Let $\mathcal{B}$ be a collection of subsets of X which satisfies:

(1) If $x \in X$ then there exists $B \in \mathcal{B}$ such that $x \in B$

(2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$

Let $\mathcal{T}$ be the minimal topology on X containing $\mathcal{B}$. Let

$$\mathcal{T}_1 = \bigcap_{\substack{\mathcal{Q} \text{ topology} \\ \mathcal{Q} \supseteq \mathcal{B}}} \mathcal{Q}$$

$$\mathcal{T}_2 = \left\{ U \subseteq X \;\middle|\; \begin{array}{l} \text{there exists } \mathcal{C} \subseteq \mathcal{B} \\ \text{with } U = \bigcup_{B \in \mathcal{C}} B \end{array} \right\}$$

$$\mathcal{T}_3 = \left\{ V \subseteq X \;\middle|\; \begin{array}{l} \text{if } x \in V \text{ then there exists} \\ B \in \mathcal{B} \text{ with } x \in B \text{ and } B \subseteq V \end{array} \right\}$$

Then $\mathcal{T} = \mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3$.

Proof

(a) Since

$$\mathcal{T}_3 = \{V \subseteq X \mid V = \bigcup_{\substack{B \in \mathcal{B} \\ B \subseteq V}} B\} \text{ then } \mathcal{T}_3 \subseteq \mathcal{T}_2$$

If $\mathcal{C} \subseteq \mathcal{B}$ and $U = \bigcup_{B \in \mathcal{C}} B$ then $U \in \mathcal{T}_3$

So $\mathcal{T}_2 \subseteq \mathcal{T}_3$, which gives $\mathcal{T}_2 = \mathcal{T}_3$.

(b) $\mathcal{T}_2$ is closed under unions.

By condition (2) on $\mathcal{B}$, $\mathcal{T}_3$ is closed under finite intersections.

By condition (1) on $\mathcal{B}$,

$$X = \bigcup_{B \in \mathcal{B}} B \quad \text{and} \quad \phi = \bigcup_{B \in \phi} B$$

so that $\phi, X \in \mathcal{T}_2$.

So $\mathcal{T}_3 = \mathcal{T}_2$ is a topology and $\mathcal{T}_3 \supseteq \mathcal{B}$.

(c) $\mathcal{T}_1 \neq \phi$ since the discrete topology ~~contains~~ on X contains $\mathcal{B}$

$\mathcal{T}_1$ is closed under unions and finite intersections since each topology Q is closed under unions and finite intersections

So $\mathcal{T}_1$ is a topology and $\mathcal{T}_1 \supseteq \mathcal{B}$.

(d) Using the definition of $\mathcal{T}_1$,

$$\mathcal{T}_2 = \mathcal{T}_3 \supseteq \mathcal{T}_1 \quad \text{and} \quad \mathcal{T} \supseteq \mathcal{T}_1$$

Using the definition of $\mathcal{T}$,

$$\mathcal{T}_1 \supseteq \mathcal{T} \quad \text{and} \quad \mathcal{T}_3 \supseteq \mathcal{T}.$$

So

$$\mathcal{T}_1 \supseteq \mathcal{T} \supseteq \mathcal{T}_3 = \mathcal{T}_2 \supseteq \mathcal{T}_1. //$$

<u>Generating sets for topologies</u>

<u>Proposition</u> Let X be a set.
Let $\mathcal{M}$ be a collection of subsets of X.
For $M \in \mathcal{M}$ let $\mathcal{T}_M = \{\emptyset, M, X\}$.
Let

$$\mathcal{B} = \{M_1 \cap \cdots \cap M_\ell \mid \ell \in \mathbb{Z}_{>0},\ M_1, M_2, \ldots, M_\ell \in \mathcal{M}\} \cup \{X\}.$$

Let $\mathcal{T}$ be the minimal topology containing $\mathcal{M}$.

$$\mathcal{T}_1 = \sup\{\mathcal{T}_M \mid M \in \mathcal{M}\}, \quad \mathcal{T}_2 = \bigcap_{\substack{\mathcal{Q} \text{ topology} \\ \mathcal{Q} \supseteq \mathcal{M}}} \mathcal{Q}$$

Then $\mathcal{B}$ satisfies conditions (1) and (2) and

$\mathcal{T} = \mathcal{T}_1 = \mathcal{T}_2$ is the minimal topology containing $\mathcal{B}$.

Proof

(a) Since topologies are closed under finite intersections then
if $\mathcal{Q}$ is a topology that contains $\mathcal{M}$
then $\mathcal{Q}$ contains $\mathcal{B}$.

(b) Since $\mathcal{B} \supseteq \mathcal{M}$ then if $\mathcal{Q}$ is a topology that contains $\mathcal{B}$ then $\mathcal{Q}$ contains $\mathcal{M}$.

Thus the minimal topology containing $\mathcal{M}$ is the minimal topology containing $\mathcal{B}$.
The conversions (a) and (b) also give

$$\mathcal{T}_2 = \bigcap_{\substack{\mathcal{Q} \text{ topology} \\ \mathcal{Q} \supseteq \mathcal{M}}} \mathcal{Q} = \bigcap_{\substack{\mathcal{Q} \text{ top} \\ \mathcal{Q} \supseteq \mathcal{B}}} \mathcal{Q}.$$

Since $X \in \mathcal{B}$ then $\mathcal{B}$ satisfies condition (1) of the previous proposition.
Since $\mathcal{B}$ is closed under finite intersections $\mathcal{B}$ satisfies condition (2) of the previous proposition.

If $M \subseteq X$ then $\mathcal{T}_M = \{\phi, M, X\}$ is a topology.
By definition $\mathcal{T}_1 = \sup\{\mathcal{T}_M \mid M \in \mathcal{M}\}$

is a topology such that

if $M \in \mathcal{M}$ then $\mathcal{T}_1 \supseteq \mathcal{T}_M$

and if $\mathcal{T}'$ is a topology on X with $\mathcal{T}' \supseteq \mathcal{T}_M$ for $M \in \mathcal{M}$ then $\mathcal{T}' \supseteq \mathcal{T}_1$

thus $\mathcal{T}_2 \subseteq \mathcal{T}_1$ by definition of $\mathcal{T}_2$

and $\mathcal{T}_2 \supseteq \mathcal{T}_1$ since $\mathcal{T}_1$ is a <u>least</u> upper bound of topologies containing $\{\mathcal{T}_M \mid M \in \mathcal{M}\}$

so $\mathcal{T}_2 = \mathcal{T}_1 = \mathcal{T}$. //