Compactness exercises

Proposition Let $(X, \mathcal{T}_X)$ be a topological space.
Let $A \subseteq X$.
If X is Hausdorff and A is compact then A is closed in X.

Proof Assume X is Hausdorff and A is compact.

To show: A is closed in X.

To show: A^c is open in X.
To show: If $x \in A^c$ then x is an interior point of A^c.
To show: There exists $U \in \mathcal{T}_X$ such that
$x \in U$ and $U \subseteq A^c$.

Assume $x \in A^c$.
Using that X is Hausdorff,
for $y \in A$ let $U_{xy}, V_{xy} \in \mathcal{T}_X$ be such that
$x \in U_{xy}$ and $y \in V_{xy}$ and $U_{xy} \cap V_{xy} = \emptyset$.

Then
$\mathcal{S} = \{V_{xy} \mid y \in A\}$ is an open cover of A.

Since A is compact there exists $\ell \in \mathbb{Z}_{>0}$ and
$y_1, \ldots, y_\ell \in A$ with $A \subseteq V_{xy_1} \cup \cdots \cup V_{xy_\ell}$.

Let $U = U_{xy_1} \cap \cdots \cap U_{xy_\ell}$.

Then $x \in U$ and
$$U \cap A = U_{xy_1} \cap \cdots \cap U_{xy_\ell} \cap (V_{xy_1} \cup \cdots \cup V_{xy_\ell}) = \emptyset.$$

So $U \in \mathcal{T}_X$ and $x \in U$ and $U \subseteq A^c$.

So x is an interior point of A^c.

So A^c is open.

So A is closed. //

<u>Proposition</u> Let $(X, \mathcal{T}_X)$ be a topological space. Let $B \subseteq A \subseteq X$.

If A is compact and B is closed in X then B is compact.

<u>Proof</u> Assume A is compact, B is closed in X and $B \subseteq A$.

To show: B is compact.

Let $\mathcal{S}$ be an open cover of B.

Since B is closed then B^c is open.

So $\mathcal{S} \cup \{B^c\}$ is an open cover of A.

Since A is compact there exist $\ell \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_\ell \in \mathcal{S}$ such that

$$A \subseteq U_1 \cup \cdots \cup U_\ell \quad \text{or} \quad A \subseteq U_1 \cup \cdots \cup U_\ell \cup B^c.$$

Since $B \subseteq A$ then

$$B \subseteq U_1 \cup \cdots \cup U_\ell. //$$

Example of a compact subset of $(X, \mathcal{T}_X)$ that is not closed:

$$\{0,\} \cup \mathbb{R}_{(0,1]} \text{ in the space of}$$

Assignment 3 question 3.

Proposition Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f: X \to Y$ be a continuous function. Let $E \subseteq X$.

If E is compact then $f(E)$ is compact.

Proof To show: If $\mathcal{S}$ is an open cover of $f(E)$ then $\mathcal{S}$ contains a finite subcover.

Assume $\mathcal{S}$ is an open cover of $f(E)$.

Then $\mathcal{R} = \{f^{-1}(V) \mid V \in \mathcal{S}\}$ is an open cover of E.

So there exist $\ell \in \mathbb{Z}_{>0}$ and $V_1, \ldots, V_\ell \in \mathcal{S}$ with

$$E \subseteq f^{-1}(V_1) \cup \cdots \cup f^{-1}(V_\ell).$$

So $f(E) \subseteq V_1 \cup \cdots \cup V_\ell$.

So $\mathcal{S}$ contains a finite subcover of $f(E)$.

So $f(E)$ is compact.

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<u>Proposition</u>

Let (X, d_X) and (Y, d_Y) be metric spaces and ~~let~~ let $f: X \to Y$ be a continuous function. If X is compact then

$f: X \to Y$ is uniformly continuous.

<u>Proof</u>

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x, a \in X$ ~~then~~ and $d_X(x,a) < \delta$ then $d(f(x), f(a)) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Let $\delta_a \in \mathbb{R}_{>0}$ be such that $B_{\delta_a}(a) \subseteq B_{\frac{\varepsilon}{10}}(f(a))$

Then $\mathcal{S} = \{ B_{\frac{\delta_a}{10}}(a) \mid a \in X \}$ is an open cover of X.

Let $\ell \in \mathbb{Z}_{>0}$ and $a_1, \ldots, a_\ell \in X$ such that

$$X = B_{\frac{\delta_{a_1}}{10}}(a_1) \cup \cdots \cup B_{\frac{\delta_{a_\ell}}{10}}(a_\ell)$$

Let $\delta = \min\{\delta_{a_1}, \ldots, \delta_{a_\ell}\}$.

Then $\delta \in \mathbb{R}_{>0}$.

To show: If $x, a \in X$ and $d(x,a) < \delta$ then $d(f(x), f(a)) < \varepsilon$.

Assume $x, a \in X$ and $(x,a) \in B_\delta$.

x a
a_i a_j

Then $x \in B_{\frac{\delta_{a_i}}{10}}(a_i)$ and $a \in B_{\frac{\delta_{a_j}}{10}}(a_j)$

So x and a_i and a are all in $B_\delta(a_j)$

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So $f(x)$ and $f(a_i)$ and $f(a_i)$ are all in $B_{\frac{\varepsilon}{10}}(a_j)$

So $f(x) \in B_\varepsilon(f(a))$.

So f is uniformly continuous.//

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