

Compactness exercises

12.10.2022
MHS Lect 31 ①

Proposition Let (X, \mathcal{T}_X) be a topological space
Let $A \subseteq X$.

If X is Hausdorff and A is compact
then A is closed in X .

Proof Assume X is Hausdorff and A is compact
To show: A is closed in X .

To show: A^c is open in X .

To show: If $x \in A^c$ then x is an interior point of A^c

To show: There exists $U \in \mathcal{T}_X$ such that
 $x \in U$ and $U \subseteq A^c$

Assume $x \in A^c$.

Using that X is Hausdorff,

for $y \in A$ let $U_{xy}, V_{xy} \in \mathcal{T}_X$ be such that

$x \in U_{xy}$ and $y \in V_{xy}$ and $U_{xy} \cap V_{xy} = \emptyset$

Then

$\mathcal{S} = \{V_{xy} \mid y \in A\}$ is an open cover of A

Since A is compact there exists $k \in \mathbb{Z}_{>0}$ and

$y_1, \dots, y_k \in A$ with $A \subseteq V_{xy_1} \cup \dots \cup V_{xy_k}$.

Let $U = U_{xy_1} \cap \dots \cap U_{xy_k}$

Then $x \in U$ and

$U \cap A = U_{xy_1} \cap \dots \cap U_{xy_k} \cap (V_{xy_1} \cup \dots \cup V_{xy_k}) = \emptyset$.

12.10.2022 (2)

MA 5 lect 32

A. Ram

So $U \in \mathcal{I}_x$ and $x \in U$ and $U \subseteq A^c$.

So x is an interior point of A^c .

So A^c is open.

So A is closed. \square

Proposition Let (X, \mathcal{I}_X) be a topological space
Let $B \subseteq A \subseteq X$.

If A is compact and B is closed in X then
 B is compact.

Proof Assume A is compact, B is closed in X
and $B \subseteq A$.

To show: B is compact.

Let \mathcal{S} be an open cover of B .

Since B is closed then B^c is open.

So $\mathcal{S} \cup \{B^c\}$ is an open cover of A .

Since A is compact there exist $k \in \mathbb{Z}_{>0}$ and

$U_1, \dots, U_k \in \mathcal{S}$ such that

$A \subseteq U_1 \cup \dots \cup U_k$ or $A \subseteq U_1 \cup \dots \cup U_k \cup B^c$.

Since $B \subseteq A$ then

$B \subseteq U_1 \cup \dots \cup U_k$. \square

Example of a compact subset of (X, τ_X)
that is not closed:

$\{0, 1\} \cup \mathbb{R}_{(0,1)}$ in the space of
Assignment 3 question 3.

Proposition Let (X, τ_X) and (Y, τ_Y) be topological
spaces and let $f: X \rightarrow Y$ be a continuous
function. Let $E \subseteq X$.

If E is compact then $f(E)$ is compact.

Proof To show: If \mathcal{S} is an open cover of $f(E)$
then \mathcal{S} contains a finite subcover.

Assume \mathcal{S} is an open cover of $f(E)$.

Then $\mathcal{R} = \{f^{-1}(V) \mid V \in \mathcal{S}\}$ is an open
cover of E .

So there exist $l \in \mathbb{Z}_{>0}$ and $V_1, \dots, V_l \in \mathcal{S}$ with
 $E \subseteq f^{-1}(V_1) \cup \dots \cup f^{-1}(V_l)$.

So $f(E) \subseteq V_1 \cup \dots \cup V_l$.

So \mathcal{S} contains a finite subcover of $f(E)$.

So $f(E)$ is compact.

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \rightarrow Y$ be a continuous function. If X is compact then

$f: X \rightarrow Y$ is uniformly continuous.

Proof

To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x, a \in X$ and $d_X(x, a) < \delta$ then $d(f(x), f(a)) < \epsilon$.

Assume $\epsilon \in \mathbb{R}_{>0}$.

Let $\delta_a \in \mathbb{R}_{>0}$ be such that $B_{\delta_a}(a) \subseteq B_{\frac{\epsilon}{10}}(f(a))$

then $\mathcal{S} = \{ B_{\frac{\delta_a}{10}}(a) \mid a \in X \}$ is an open cover of X .

Let $\mathcal{L} \in \mathcal{L}_{>0}$ and $a_1, \dots, a_\ell \in X$ such that

$$X = B_{\frac{\delta_{a_1}}{10}}(a_1) \cup \dots \cup B_{\frac{\delta_{a_\ell}}{10}}(a_\ell)$$

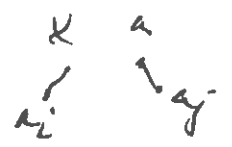
Let $\delta = \min \{ \delta_{a_1}, \dots, \delta_{a_\ell} \}$.

then $\delta \in \mathbb{R}_{>0}$.

To show: If $x, a \in X$ and $d(x, a) < \delta$ then $d(f(x), f(a)) < \epsilon$.

Assume $x, a \in X$ and $(x, a) \in B_\delta$.

Then $x \in B_{\frac{\delta_{a_i}}{10}}(a_i)$ and $a \in B_{\frac{\delta_{a_j}}{10}}(a_j)$



So x and a_i and a are all in $B_\delta(a_j)$

So $f(x)$ and $f(a_i)$ and $f(a_j)$ are all in $B_{\frac{\epsilon}{10}}(a_j)$ ^{12.10.2022} (5)

So $f(x) \in B_{\frac{\epsilon}{10}}(f(a_j))$.

So f is uniformly continuous. //

MH5lect32

A. Ram