Duals

Let $\mathbb{F}$ be a field

Let V be an $\mathbb{F}$-vector space.

The <u>dual of V</u> is the $\mathbb{F}$-vector space

$$V^* = \mathrm{Hom}(V, \mathbb{F})$$

$$= \{\varphi : V \to \mathbb{F} \mid \varphi \text{ is a linear transformation}\}$$

with addition and scalar multiplication given by

$$(\varphi_1 + \varphi_2)(v) = \varphi_1(v) + \varphi_2(v), \quad \text{for } v \in V,\ \varphi_1, \varphi_2 \in V^*,$$

$$(c\varphi)(v) = c\varphi(v), \quad \text{for } c \in \mathbb{F},\ \varphi \in V^* \text{ and } v \in V.$$

<u>HW</u>: Show that V^* satisfies the vector space conditions.

Let V and W be $\mathbb{F}$-vector spaces and let

$$T : V \to W$$

be a linear transformation.

The <u>adjoint of T</u> is the linear transformation

$$T^* : W^* \to V^*$$

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ & & \downarrow \psi \\ & & \mathbb{F} \end{array}$$

given by

$$(T^* \psi)(v) = \psi(T(v)), \quad \text{for } v \in V,\ \psi \in W^*$$

HW: Show that T^* satisfies the linear transformation conditions.

Evaluations

For $v \in V$ define

$$ev_v : V^* \longrightarrow \mathbb{F} \text{ by } ev_v(\varphi) = \varphi(v),$$

for $\varphi \in V^*$.

In other words, define a function

$$ev : V \longrightarrow (V^*)^*$$

$$v \longmapsto ev_v : V^* \longrightarrow \mathbb{F}$$

$$\varphi \longmapsto \varphi(v)$$

Claim: ev is an injective linear transformation

Proof sketch:

If $v_1, v_2 \in V$ and $\varphi \in V^*$ then

$$ev_{v_1+v_2}(\varphi) = \varphi(v_1+v_2) = \varphi(v_1) + \varphi(v_2)$$

$$= ev_{v_1}(\varphi) + ev_{v_2}(\varphi),$$

so that $ev_{v_1+v_2} = ev_{v_1} + ev_{v_2}$.

If $v \in V$, $c \in \mathbb{F}$ and $\varphi \in V^*$ then

$$ev_{cv}(\varphi) = \varphi(cv) = c\varphi(v) = c\,ev_v(\varphi)$$

so that $ev_{cv} = c\,ev_v$.

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The kernel of ev is

$$\ker(ev) = \{ v \in V \mid ev_v = 0 \}$$

$$= \left\{ v \in V \;\middle|\; \begin{array}{l} \text{if } \varphi \in V^* \text{ then} \\ ev_v(\varphi) = 0 \end{array} \right\}$$

$$= \{ v \in V \mid \text{if } \varphi \in V^* \text{ then } \varphi(v) = 0 \}.$$

If $v \in V$ and $v \neq 0$ then there is a basis B of V with $v \in B$ and the linear transformation $\gamma : V \to \mathbb{F}$ given by $\gamma(v) = 1$ and

$\gamma(b) = 0$ for $b \in B$ with $b \neq v$,

has $ev_v(\gamma) = 1 \neq 0$.

So, if $v \in V$ satisfies

if $\varphi \in V^*$ then $\varphi(v) = 0$

then $v = 0$. So

$$\ker(ev) = 0.$$

So ev is injective.

Since $ev : V \to (V^*)^*$ is injective we might abuse notation and write

$$V \subseteq (V^*)^*.$$

Let V be an $\mathbb{F}$-vector space.

A <u>bilinear form on V</u> is a function

$$\begin{aligned}\langle\,,\,\rangle : V\times V &\longrightarrow \mathbb{F}\\ (v_1, v_2) &\longmapsto \langle v_1, v_2\rangle\end{aligned}$$

such that

(a) If $w, v_1, v_2 \in V$ and $c_1, c_2 \in \mathbb{F}$ then

$$\langle c_1v_1 + c_2v_2, w\rangle = c_1\langle v_1, w\rangle + c_2\langle v_2, w\rangle.$$

(b) If $v, w_1, w_2 \in V$ and $c_1, c_2 \in \mathbb{F}$ then

$$\langle v, c_1w_1 + c_2w_2\rangle = c_1\langle v, w_1\rangle + c_2\langle v, w_2\rangle.$$

<u>The data of $\langle\,,\,\rangle$ is the same as the data of a linear transformation $V \to V^*$</u>

(a) Let $\langle\,,\,\rangle : V\times V \to \mathbb{F}$ be a bilinear form.

Define

$$\begin{aligned}\gamma : V &\to V^*\\ v &\mapsto \gamma_v\end{aligned}$$ by setting $\gamma_v(x) = \langle v, x\rangle$

for $x \in V$. Then $\gamma : V \to V^*$ is a linear transformation.

Let

$$\begin{aligned}\Phi : V &\to V^*\\ v &\mapsto \varphi_v\end{aligned}$$ be a linear transformation.

Let $\langle\, ,\rangle : V \times V \to \mathbb{F}$ be a bilinear form on V MHSLect31

The bilinear form $\langle\, ,\rangle$ is <u>nondegenerate</u> if

$\langle\, ,\rangle$ satisfies:

if $v \in V$ and $v \neq 0$ then there exists

$x \in V$ such that $\langle x, v\rangle \neq 0$.

So $\langle\, ,\rangle$ is nondegenerate means

if $v \in V$ and $v \neq 0$ then $\gamma_v \neq 0$.

So $\langle\, ,\rangle$ is nondegenerate means $\ker(\gamma) = 0$.

So $\langle\, ,\rangle$ is nondegenerate means γ is injective.