07.10.2022

MHS Lect 30 ①

Filters

We had three kinds of limits already:

$$\lim_{x\to a} f(x), \qquad \lim_{\substack{x\to a\\ x\neq a}} f(x), \qquad \lim_{n\to\infty} a_n$$

Are there other kinds of limits?

Let X be a set. A <u>filter</u> on X is a collection $\mathcal{F}$ of subsets of X such that

(a) $\emptyset \notin X$

(b) If $N \in \mathcal{F}$ and $P \subseteq X$ and $P \supseteq N$ then $P \in \mathcal{F}$

(c) If $\ell \in \mathbb{Z}_{>0}$ and $N_1, N_2, \ldots, N_\ell \in \mathcal{F}$ then $N_1 \cap \cdots \cap N_\ell \in \mathcal{F}$.

Let $(X, \mathcal{T}_X)$ be a topological space.

Let $\mathcal{F}$ be a filter on X.

A <u>limit point of $\mathcal{F}$</u> is $z \in X$ such that

$$\mathcal{F} \supseteq \mathcal{N}(z)$$

A <u>cluster point of $\mathcal{F}$</u> is $z \in X$ such that there exists a filter $\mathcal{G}$ on X with

$$\mathcal{G} \supseteq \mathcal{F} \text{ and } z \text{ is a limit point of } \mathcal{G}.$$

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Examples of filters

(1) The tail filter on $\mathbb{Z}_{>0}$ is

$$\mathcal{F} = \left\{ N \subseteq \mathbb{Z}_{>0} \;\middle|\; \begin{array}{l} N \text{ contains all but a} \\ \text{finite number of elements} \\ \text{of } \mathbb{Z}_{>0} \end{array} \right\}$$

$\mathcal{F}$ is the smallest filter containing

$$\mathcal{B} = \{ \mathbb{Z}_{\geq n} \mid n \in \mathbb{Z}_{\geq 0} \}.$$

(2) Let $(Y, \mathcal{T}_Y)$ be a topological space. Let $y \in Y$. The neighborhood filter of y is

$$\mathcal{N}(y) = \{\text{neighborhoods of } y\}$$

$$= \left\{ N \subseteq Y \;\middle|\; \begin{array}{l} \text{there exists } U \in \mathcal{T}_Y \text{ with} \\ y \in U \text{ and } U \subseteq N \end{array} \right\}$$

$\mathcal{F}$ is the smallest filter containing

$$\mathcal{B}(y) = \{ U \subseteq Y \mid U \in \mathcal{T}_Y \text{ and } y \in U \}.$$

(3) Let $(X, \mathcal{F}_X)$ be a filtered space and let $(Y, \mathcal{T}_Y)$ be a topological space.

~~The~~ Let $f: X \to Y$ be a function.

The image filter is the filter on Y given by

$$f(\mathcal{F}_X) = \{ f(N) \mid N \in \mathcal{F}_X \}$$

Write

$$z = \lim_{\mathcal{F}_x} f$$

if z is a limit point of $f(\mathcal{F}_x)$.

<u>Proposition</u> Let $(X, \mathcal{T}_X)$ be a topological space and let $\mathcal{F}$ be a filter on X.

(a) If z is a limit point of $\mathcal{F}$ then
z is a cluster point of $\mathcal{F}$.

(b) An element $z \in X$ is a cluster point of $\mathcal{F}$ if and only if

$$z \in \bigcap_{N \in \mathcal{F}} \overline{N}$$

<u>Definition</u> An <u>ultrafilter on X</u> is a filter $\mathcal{F}$ on X such that every cluster point of $\mathcal{F}$ is a limit point of $\mathcal{F}$.

Part (b) of the proposition says that

$$\{\text{cluster points of } \mathcal{F}\} = \bigcap_{N \in \mathcal{F}} \overline{N}.$$

Proof of the Proposition

Let $R = \{\text{cluster points of } \mathcal{F}\}$.

To show: (a) $\left(\bigcap_{N \in \mathcal{F}} \bar{N}\right) \subseteq R$

(b) $R \subseteq \left(\bigcap_{N \in \mathcal{F}} \bar{N}\right)$

(a) Let $z \in \bigcap_{N \in \mathcal{F}} \bar{N}$

If $U \in \mathcal{N}(z)$ and $N \in \mathcal{F}$ then $U \cap N \neq \phi$.
Hence there exists a filter $\mathcal{G}$ generated by $\mathcal{N}(z)$ and $\mathcal{F}$.

Since $\mathcal{G} \supseteq \mathcal{N}(z)$ then z is a limit point of $\mathcal{G}$.
Since $\mathcal{G} \supseteq \mathcal{F}$ then z is a cluster point of $\mathcal{F}$.
So $\left(\bigcap_{N \in \mathcal{F}} \bar{N}\right) \subseteq R$.

(b) Let $z \in R$.
Since z is a cluster point of $\mathcal{F}$ then there exists $\mathcal{G} \supseteq \mathcal{F}$ with $\mathcal{G} \supseteq \mathcal{N}(z)$.
Since $\mathcal{G}$ is a filter and $\mathcal{G} \supseteq \mathcal{N}(z)$ and $\mathcal{G} \supseteq \mathcal{F}$ then if $U \in \mathcal{N}(z)$ and $N \in \mathcal{F}$ then $U \cap N \neq \phi$.
So z is a close point to N.
So $z \in \bar{N}$ and $z \in \left(\bigcap_{N \in \mathcal{F}} \bar{N}\right)$.
So $R = \left(\bigcap_{N \in \mathcal{F}} \bar{N}\right)$ //.