Properties of $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$

Theorem

(a) $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ are Hausdorff

(b) $\mathbb{R}_{\geq 0}$ is complete

(c) $\mathbb{R}_{\geq 0}$ is locally compact.

(d) $\mathbb{R}_{\geq 0}$ is not compact.

Theorem Let $A \subseteq \mathbb{R}$.

(a) A is connected if and only if A is an interval

(b) A is compact if and only if A is closed and bounded.

Theorem (Least upper bound property)
If $A \subseteq \mathbb{R}$ and $A \neq \emptyset$ and A is bounded then $\sup(A)$ exists in $\mathbb{R}$.

Theorem If $(a_1, a_2, \ldots)$ is an increasing bounded sequence in $\mathbb{R}_{\geq 0}$ then $(a_1, a_2, \ldots)$ converges in $\mathbb{R}$ to $\sup\{a_1, a_2, \ldots\}$.

Theorem Let $n \in \mathbb{Z}_{>0}$. The function $x^n: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous, bijective and monotone.

Theorem The function $e^x: \mathbb{R} \to \mathbb{R}_{>0}$ is continuous bijective and monotone.

Theorem Let $\iota: \mathbb{Q} \to \mathbb{R}$ be the inclusion.

(a) ι is injective and is a field homomorphism

(b) ι is not surjective.

(c) $\overline{\mathbb{Q}} = \mathbb{R}$

(d) $\hat{\mathbb{Q}} \cong \mathbb{R}$

(e) If $x, y \in \mathbb{R}$ and $x < y$ then there exists $c \in \mathbb{Q}$ with $x < c < y$.

(f) If $x, y \in \mathbb{R}$ and $x < y$ then there exists $c \in \mathbb{R} - \mathbb{Q}$ with $x < c < y$.

Theorem $\mathbb{R}$ with $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ given by

$$d(x, y) = |y - x|$$

is a metric space (hence a uniform space and a topological space).

<u>Theorem</u> $\mathbb{R}$ is an ordered field.

and $\mathbb{Q}$ is an ordered field.

An <u>ordered field</u> is a set $\mathbb{F}$ with functions

$$\begin{matrix} \mathbb{F}\times\mathbb{F} \to \mathbb{F} \\ (a,b) \mapsto a+b \end{matrix} \quad \text{and} \quad \begin{matrix} \mathbb{F}\times\mathbb{F} \to \mathbb{F} \\ (a,b) \mapsto ab \end{matrix}$$

~~such that~~ and a relation $\le$ such that

(a) If $a,b,c \in \mathbb{F}$ then $(a+b)+c = a+(b+c)$.

(b) If $a,b \in \mathbb{F}$ then $a+b = b+a$

(c) There exists $0 \in \mathbb{F}$ which satisfies

if $x \in \mathbb{F}$ then $0+x = x$ and $x+0 = x$

(d) If $a \in \mathbb{F}$ then there exists $-a \in \mathbb{F}$ such that

$a+(-a) = 0$ and $(-a)+a = 0$.

(e) If $a,b,c \in \mathbb{F}$ then $(ab)c = a(bc)$.

(f) If $a,b \in \mathbb{F}$ then $ab = ba$

(g) There exists $1 \in \mathbb{F}$ which satisfies

if $x \in \mathbb{F}$ then $1\cdot x = x$ and $x \cdot 1 = x$.

(h) If $a \in \mathbb{F}$ and $a \ne 0$ then there exists $a^{-1} \in \mathbb{F}$ such that $aa^{-1} = 1$ and $a^{-1}a = 1$.

(i) If $a,b,c \in \mathbb{F}$ then

$$a(b+c) = ab+ac \quad \text{and} \quad (a+b)c = ac+bc$$

(j) If $a,b \in \mathbb{F}$ then $a \le b$ or $b \le a$

(k) If $a,b \in \mathbb{F}$ and $a \le b$ and $b \le a$ then $b = a$.

(l) If $a,b,c \in \mathbb{F}$ and $a \le b$ and $b \le c$ then $a \le c$.

(m) If $a,b,c \in \mathbb{F}$ and $a \le b$ then

$$a+c \le b+c.$$

(n) If $a,b \in \mathbb{F}$ and $a \ge 0$ and $b \ge 0$ then

$$ab \ge 0.$$

An <u>ordered topological field</u> is a field $\mathbb{F}$ with a topology $\mathcal{T}_{\mathbb{F}}$ such that

$$\begin{matrix} \mathbb{F} \times \mathbb{F} \to \mathbb{F} \\ (a,b) \mapsto a+b \end{matrix} \quad \text{and} \quad \begin{matrix} \mathbb{F} \times \mathbb{F} \to \mathbb{F} \\ (a,b) \mapsto ab \end{matrix}$$

are continuous.

<u>Theorem</u> $\mathbb{R}$ and $\mathbb{Q}$ are ordered topological fields.

<u>Theorem</u> $\mathbb{C}$ is a topological field but it is <u>not</u> ordered.