Topological and Uniform spaces form categories

Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be uniform spaces.
A function $f: X \to Y$ is <u>uniformly</u> <u>continuous</u> if f satisfies:

$$\text{if } E \in \mathcal{E}_Y \text{ then } (f\times f)^{-1}(E) \in \mathcal{E}_X$$

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.
A function $f: X \to Y$ is <u>continuous</u> if f satisfies:

$$\text{if } V \in \mathcal{T}_Y \text{ then } f^{-1}(V) \in \mathcal{T}_X.$$

<u>Proposition</u>

(a) Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ be topological spaces and let

$$f: X \to Y \quad \text{and} \quad g: Y \to Z$$

be continuous functions. Then

$$g\circ f: X \to Z \text{ is continuous.}$$

(b) Let $(X, \mathcal{E}_X)$, $(Y, \mathcal{E}_Y)$ and $(Z, \mathcal{E}_Z)$ be uniform spaces and let

$$f: X \to Y \quad \text{and} \quad g: Y \to Z$$

be uniformly continuous functions. Then

$$g\circ f: X \to Z \text{ is uniformly continuous.}$$

Every uniform space is almost a metric space

Let $(X, \mathcal{E}_X)$ be a uniform space.

Let $E \in \mathcal{E}_X$.

Construction 1 starting with E

Let $D_0 = E$.

Let $E_1 = D_0 \cap \sigma(D_0)$

Let $D_1 \in \mathcal{E}_X$ be such that $D_1 \times D_1 \subseteq E_1$

Let $E_2 = D_1 \cap \sigma(D_1)$

Let $D_2 \in \mathcal{E}_X$ be such that $D_2 \times D_2 \subseteq E_2$

Let $E_3 = D_2 \cap \sigma(D_2)$

Continue this process to produce $E_1, E_2, \ldots$

Define

$$\mathcal{X}_E = \left\{ D \subseteq X \times X \;\middle|\; \begin{matrix} \text{there exists } k \in \mathbb{Z}_{>0} \\ \text{with } D \supseteq E_k \end{matrix} \right\}$$

Little proposition

$\mathcal{X}_E$ is a uniformity on X.

Construction 2 starting with E

Let $F_0 = E$

Let $U_1 = F_0 \cap \sigma(F_0)$.

Let $F_1 \in \mathcal{E}_X$ be such that $F_1 \bowtie F_1 \bowtie F_1 \subseteq U_1 \cap F_0$

Let $U_2 = F_1 \cap \sigma(F_1)$

Let $F_2 \in \mathcal{E}_X$ be such that $F_2 \bowtie F_2 \bowtie F_2 \subseteq U_2 \cap F_1$

Let $U_3 = F_2 \cap \sigma(F_2)$

Continue this process to produce $U_1, U_2, \ldots$

Define

$$\mathcal{X}_U = \left\{ D \subseteq X \times X \;\middle|\; \begin{array}{l} \text{there exists } k \in \mathbb{Z}_{>0} \\ \text{with } D \supseteq U_k \end{array} \right\}$$

Little proposition

$\mathcal{X}_U$ is a uniformity on X.

Construction 3 starting with $\mathbb{E}$

Define $g: X\times X \to \mathbb{R}_{\geq 0}$ by

$$g(x,y) = \begin{cases} 1, & \text{if } (x,y)\notin \mathcal{U}_1, \\ 2^{-k}, & \text{if } (x,y)\in \mathcal{U}_1, \ldots, (x,y)\in \mathcal{U}_k,\ (x,y)\notin \mathcal{U}_{k+1}, \\ 0, & \text{if } (x,y)\in \mathcal{U}_k \text{ for } k\in \mathbb{Z}_{>0}. \end{cases}$$

Define $d: X\times X \to \mathbb{R}_{\geq 0}$ by

$$d(x,y) = \inf\left\{ g(x,z_1)+g(z_1,z_2)+\cdots+g(z_{p-1},y) \;\middle|\; \begin{matrix} p\in \mathbb{Z}_{>0} \\ z_1,\ldots,z_p\in X \\ z_p = y \end{matrix} \right\}$$

Define

$$\mathcal{X}_d = \left\{ D\subseteq X\times X \;\middle|\; \begin{matrix}\text{there exists } \varepsilon\in \mathbb{E} \\ \text{with } D \supseteq B_\varepsilon \end{matrix}\right\}$$

where $B_\varepsilon = \{(x,y) \mid d(x,y) < \varepsilon\}$.

<u>Proposition</u> $d: X\times X \to \mathbb{R}_{\geq 0}$ satisfies

(a) if $x\in X$ then $d(x,x) = 0$,

(b) if $x,y\in X$ then $d(x,y) = d(y,x)$,

(c) if $x,y,z\in X$ then

$$d(x,y) \leq d(x,z) + d(z,y)$$

Theorem $\mathcal{X}_E = \mathcal{X}_u = \mathcal{X}_d$

Theorem $\mathcal{E}_X = \sup\{\mathcal{X}_E \mid E \in \mathcal{E}_X\}$.

Definitions

If $E \subseteq X \times X$ then

$$\sigma(E) = \{(y,x) \mid (x,y) \in E\} \quad \text{and}$$

$$E \bowtie E = \{(x,y) \mid \text{there exists } z \in X \text{ with } (x,z) \in E \text{ and } (z,y) \in E\}$$

Let (X, d_X) be a metric space.
The <u>metric space uniformity on X</u> is

$$\mathcal{E}_X = \left\{ D \subseteq X \times X \;\middle|\; \begin{array}{l} \text{there exists } \varepsilon \in E \\ \text{with } D \supseteq B_\varepsilon \end{array} \right\}$$

where $B_\varepsilon = \{(x,y) \mid d_X(x,y) < \varepsilon\}$.