Function spaces

Favourite example of a Banach space:

$\mathbb{R}^2$ with $\| \ \| : \mathbb{R} \to \mathbb{R}_{\geq 0}$ given by

$$\|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2}$$

Let $W = \{ f : \{1, 2\} \to \mathbb{R} \}$. Then

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow W \\ x = (x_1, x_2) &\longmapsto f_x : \{1, 2\} \to \mathbb{R} \\ & \qquad\quad 1 \mapsto x_1 \\ & \qquad\quad 2 \mapsto x_2 \end{aligned}$$

is a bijection

and we can identify $\mathbb{R}^2$ and W.

<u>HW</u>: Prove that $\mathbb{R}^2$ is a Banach space.

Let V be a set.

A <u>sequence in V</u> is a function

$$\begin{aligned} x : \mathbb{Z}_{>0} &\longrightarrow V \\ 1 &\longmapsto x_1 \\ 2 &\longmapsto x_2 \\ &\vdots \end{aligned}$$

So $(x_1, x_2, \ldots)$ is a function.

Let $n \in \mathbb{Z}_{>0}$. A <u>n-tuple</u> $x = (x_1, x_2, \ldots x_n)$ in $\mathbb{R}^n$ is the same as a function $x : \{1, 2, \ldots, n\} \to \mathbb{R}$.

$\mathbb{R}^\infty = \{f : \mathbb{Z}_{>0} \to \mathbb{R}\} = \{\text{sequences in } \mathbb{R}\}$.

Define <u>addition</u> $\mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$

and <u>scalar multiplication</u> $\mathbb{R} \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots)$$

(or equivalently $(x+y)(i) = x(i) + y(i)$)

$$c(x_1, x_2, \ldots) = (cx_1, cx_2, \ldots)$$

(or equivalently $(cx)(i) = cx(i)$).

Then $\mathbb{R}^\infty$ is an $\mathbb{R}$-vector space.

Try to define norms on $\mathbb{R}^\infty$.

$$\|(x_1, x_2, \ldots)\|_2 = \left(\sum_{i=1}^{\infty} x_i^2\right)^{1/2}$$

$$\|(x_1, x_2, \ldots)\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}, \quad \text{for } p \in \mathbb{R}_{>1}.$$

$$\|(x_1, x_2, \ldots)\|_1 = \sum_{i=1}^{\infty} |x_i|$$

$$\|(x_1, x_2, \ldots)\|_\infty = \sup\{|x_1|, |x_2|, \ldots\}.$$

These <u>don't</u> satisfy the conditions for a normed vector space.

Theorem

$c_c = \{x \in \mathbb{R}^\infty \mid x_n \text{ is eventually all } 0\text{s}\}$

$= \left\{x \in \mathbb{R}^\infty \;\middle|\; \begin{array}{l}\text{there exists } N \in \mathbb{Z}_{>0} \text{ such that} \\ \text{if } n \in \mathbb{Z}_{>N} \text{ then } x_n = 0\end{array}\right\}$ with $\| \|_1$ or $\| \|_p$ or $\| \|_\infty$

$\ell^1 = \{x \in \mathbb{R}^\infty \mid \|x\|_1 \text{ exists in } \mathbb{R}_{\geq 0}\}$ with $\| \|_1$

$\ell^p = \{x \in \mathbb{R}^\infty \mid \|x\|_p \text{ exists in } \mathbb{R}_{\geq 0}\}$ with $\| \|_p$

$\ell^\infty = \{x \in \mathbb{R}^\infty \mid \|x\|_\infty \text{ exists in } \mathbb{R}_{\geq 0}\}$ with $\| \|_\infty$

$c_0 = \{x \in \mathbb{R}^\infty \mid \lim_{n \to \infty} x_n = 0\}$ with $\| \|_\infty$

are normed vector spaces.

Question Which of these normed vector spaces are Banach spaces?

Theorem The Hölder and Minkowski inequalities. Let $p, q \in \mathbb{R}_{>1}$ with $1 < p \leq 2 \leq q$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Let $x = (x_1, x_2, \ldots) \in \ell^p$ and $y = (y_1, y_2, \ldots) \in \ell^p$.

Then

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

(b) Let $x = (x_1, x_2, \ldots) \in \ell^p$ and $y = (y_1, y_2, \ldots) \in \ell^q$ and let $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$.

Then $|\langle x, y \rangle| \leq \|x\|_p \|y\|_q$

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<u>Baby case</u>: Let $n \in \mathbb{Z}_{>0}$. ~~and~~
For $x = (x_1, x_2, \ldots x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots y_n) \in \mathbb{R}^n$
define
$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \text{ and}$$
$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$
Then
$$\|x + y\| \leq \|x\| + \|y\| \qquad \text{Triangle inequality}$$
and
$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|. \qquad \text{Cauchy-Schwarz.}$$