Uniform convergence and Pointwise convergence

Let (X, d_X) and (C, d_C) be metric spaces.

Let
$$F = \{\text{functions } f : X \to C\}$$
and define $d_\infty : F \times F \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ by
$$d_\infty(f, g) = \sup\{ d_C(f(x), g(x)) \mid x \in X\}$$

Let $(f_1, f_2, \ldots)$ be a sequence in F. Let $f \in F$.

The sequence $(f_1, f_2, \ldots)$ converges pointwise to f if the sequence $(f_1, f_2, \ldots)$ satisfies:

if $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{\geq \ell}$ then $d_C(f_n(x), f(x)) < \varepsilon$.

The sequence $(f_1, f_2, \ldots)$ converges uniformly to f if the sequence $(f_1, f_2, \ldots)$ satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that

if $x \in X$ and $n \in \mathbb{Z}_{\geq \ell}$ then $d_C(f_n(x), f(x)) < \varepsilon$.

These are equivalent to:

The sequence $(f_1, f_2, \ldots)$ <u>converges pointwise to f</u> if the sequence $(f_1, f_2, \ldots)$ satisfies:

if $x \in X$ then $\lim_{n \to \infty} d_C(f_n(x), f(x)) = 0$.

The sequence $(f_1, f_2, \ldots)$ <u>converges uniformly to f</u> if the sequence $(f_1, f_2, \ldots)$ satisfies:

$$\lim_{n \to \infty} d_\infty(f_n, f) = 0.$$

<u>Proposition</u> If $(f_1, f_2, \ldots)$ <u>converges uniformly to f</u> then $(f_1, f_2, \ldots)$ converges pointwise to f.

<u>Sample exam 4 Question 7</u>

Let $X = \mathbb{R}_{[0,1]}$ with $d_X(x,y) = |y-x|$ and $C = \mathbb{R}_{[0,1]}$ with $d_C(x,y) = |y-x|$.

For $n \in \mathbb{Z}_{>0}$ let

$$\begin{aligned} f_n : \mathbb{R}_{[0,1]} &\to \mathbb{R}_{[0,1]} \\ x &\mapsto x^n \end{aligned}$$

Let $f : \mathbb{R}_{[0,1]} \to \mathbb{R}_{[0,1]}$ be given by

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R}_{[0,1)}, \\ 1, & \text{if } x = 1. \end{cases}$$

(a) Show that $(f_1, f_2, \ldots)$ converges pointwise to f.

(b) Show that $(f_1, f_2, \ldots)$ does not converge uniformly to f.

(c) Show that $f_1, f_2, \ldots$ are all continuous.

(d) Show that f is <u>not</u> continuous.

<u>Sample exam 1 Question 6</u>

Define $f_n : [0,1] \to \mathbb{R}$ for $n \in \mathbb{Z}_{>0}$ by

$$f_n(x) = \frac{nx^n}{1+nx}, \quad \text{for } x \in [0,1]$$

Find the pointwise limit f of $(f_1, f_2, \ldots)$ and determine whether $(f_1, f_2, \ldots)$ converges uniformly to f.

<u>Sample exam 2 Question 3</u>

Let

$$f_n(x) = \frac{1-x^n}{1+x^n} \quad \text{for } x \in [0,1] \text{ and } n \in \mathbb{Z}_{>0}.$$

Find the pointwise limit f of $(f_1, f_2, \ldots)$.

Determine whether $(f_1, f_2, \ldots)$ is uniformly convergent to f or not on the interval $[0,1]$.

Is the sequence $(f_1, f_2, \ldots)$ uniformly convergent on the interval $[0,1]$?

2014 Assignment 1 Question 5

Determine whether the following sequences of functions converge uniformly.

(a) $f_n(x) = e^{-nx^2}$, $x \in [0,1]$;

(b) $g_n(x) = e^{-x^2/n}$, $x \in [0,1]$

(c) $h_n(x) = e^{-x^2/n}$, $x \in \mathbb{R}$.

Uniformly continuous functions

Let (X, d_X) and (Y, d_Y) be metric spaces.

Let $f: X \to Y$ be a function.

The function f is continuous if f satisfies:

if $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

if $y \in X$ and $d_X(x,y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$.

The function f is uniformly continuous if f satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

if $x \in X$ and $y \in X$ and $d_X(x,y) < \delta$ then $d_Y(f(y), f(x)) <$

Show that multiplication

$$\begin{aligned} \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (x,y) &\mapsto xy \end{aligned}$$

is continuous but not uniformly continuous.