Compactness: Existence of limits

Let (X, d_X) be a metric space. Let $A \subseteq X$.

$$\begin{tikzcd}
\text{cover compact} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{ball compact} \arrow[r, Rightarrow] & \text{bounded} \\
\text{sequentially compact} \arrow[r, Rightarrow] & \text{Cauchy compact} \arrow[r, Rightarrow] & \text{closed}
\end{tikzcd}$$

Cover compact

In English: Every open cover has a finite subcover.

In Math: Let $(X, \mathcal{T}_X)$ be a topological space. Let $A \subseteq X$. The set A is compact, or cover compact, if A satisfies:

if $\mathcal{S} \subseteq \mathcal{T}_X$ and $A \subseteq \left(\bigcup_{U \in \mathcal{S}} U \right)$ then there exist $\ell \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_\ell \in \mathcal{S}$ such that $A \subseteq U_1 \cup \cdots \cup U_\ell$.

Let

$$E = \{10^{-1}, 10^{-2}, 10^{-3}, \ldots\}$$

Ball compact

In English: A is covered by a finite number of ε-balls.

In Math: Let (X, d_X) be a metric space. Let $A \subseteq X$. The set A is ball compact in X, or precompact, or totally bounded, if A satisfies

If $\varepsilon \in \mathbb{E}$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in X$ such that

$$A \subseteq B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_\ell).$$

Bounded

In English: A has finite size.

In Math: Let (X, d_X) be a metric space. Let $A \subseteq X$. The set A is bounded if there exists $M \in \mathbb{R}_{>0}$ and $x \in X$ such that $A \subseteq B_M(x)$.

Philosophy

Bounded: A is covered by a single ball.

Ball compact: You choose any $\varepsilon \in \mathbb{E}$. Then A is covered by a finite number of ε-balls.

Cover compact: For any cover of A a finite number of them will do.

Sequentially compact

In English: Every sequence has some sort of a limit point.

In Math: Let $(X, \mathcal{T}_X)$ be a topological space. Let $A \subseteq X$. The set A is <u>sequentially compact</u> if A satisfies:

If $(a_1, a_2, \ldots)$ is a sequence in A then there exists $z \in A$ such that z is a cluster point of $(a_1, a_2, \ldots)$.

Cauchy compact

In English: Every Cauchy sequence in A has a limit point in A

<u>In Math</u>: Let (X, d_X) be a metric space. Let $A \subseteq X$. The set A is <u>Cauchy compact</u>, or <u>complete</u> if A satisfies:

if $(a_1, a_2, \ldots)$ is a Cauchy sequence in A then there exists $z \in A$ such that z is a limit point of $(a_1, a_2, \ldots)$.

Closed

In English: Every close point to A is in A.

In Math: Let $(X, \mathcal{T}_X)$ be a topological space. Let $A \subseteq X$. The set A is closed in X if A satisfies:

$$A^c \in \mathcal{T}_X$$

Examples

(1) Closed $\not\Rightarrow$ Cauchy compact

Let $X = \mathbb{R}_{(0,1]}$ and let $A = X$ (with the standard metric space topology from

$$d(x,y) = |y - x|.$$

Then A is closed in X and $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$ is a Cauchy sequence in A which does not converge.

(2) Cauchy compact $\not\Rightarrow$ sequentially compact.

Let $X = \mathbb{R}$ with the standard metric and let $A = X$. Then A is complete and $(1, 2, 3, \ldots)$ is a sequence in A with no cluster point.

(3) Bounded $\not\Rightarrow$ Ball compact

Let $X = \mathbb{R}$ with metric $d_X : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ given by

$$d_X(x,y) = \min\{|x-y|, 1\}.$$

Let $A = X$. Then $A \subseteq B_2(0)$ but A is not covered by a finite number of ε-balls of radius $\frac{1}{10}$.

(4) Ball compact $\not\Rightarrow$ cover compact.

Let $X = \mathbb{R}$ and let $A = \mathbb{R}_{(0,1)}$ where the metric on $\mathbb{R}$ is the standard metric. Then A is ball compact but not closed and bounded

In $\mathbb{R}^n$ (with the standard metric)

A is cover compact $\Leftrightarrow$ A is closed and bounded.