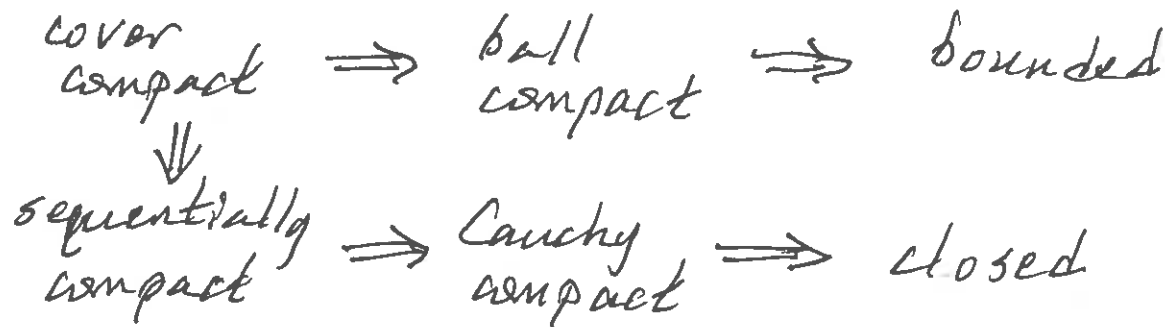


Compactness: Existence of limits

05.09.2012
MHS Lect 19 (1)

Let (X, d_X) be a metric space. Let $A \subseteq X$.



Cover compact

In English: Every open cover has a finite subcover.

In Math: Let (X, \mathcal{T}_X) be a topological space. Let $A \subseteq X$. The set A is compact, or cover compact, if A satisfies:

if $\mathcal{S} \subseteq \mathcal{T}_X$ and $A \subseteq \left(\bigcup_{U \in \mathcal{S}} U \right)$ then there exist $l \in \mathbb{N}_{>0}$ and $U_1, \dots, U_l \in \mathcal{S}$ such that $A \subseteq U_1 \cup \dots \cup U_l$.

Let

$$E = \{10^{-1}, 10^{-2}, 10^{-3}, \dots\}$$

Ball compact

In English: A is covered by a finite number of ε -balls.

In Math: Let (X, d_X) be a metric space.

Let $A \subseteq X$. The set A is ball compact in X , or precompact, or totally bounded, if A satisfies

$\forall \varepsilon \in \mathbb{R}$ then there exists $\ell \in \mathbb{Z}_{>0}$ and

$x_1, x_2, \dots, x_\ell \in X$ such that

$$A \subseteq B_\varepsilon(x_1) \cup \dots \cup B_\varepsilon(x_\ell).$$

Bounded

In English: A has finite size.

In Math: Let (X, d_X) be a metric space.

Let $A \subseteq X$. The set A is bounded if

there exists $M \in \mathbb{R}_{>0}$ and $x \in X$

such that $A \subseteq B_M(x)$.

Philosophy

Bounded: A is covered by a single ball.

Ball compact: You choose ^{any} $\varepsilon \in \mathbb{R}$. Then A is covered by a finite number of ε -balls.

Cover compact: For any cover of A a finite number of them will do.

Sequentially compact

05.09.2011 (3)
MHS Lect 19

In English: Every sequence has some sort of a limit point.

In Math: Let (X, \mathcal{T}_X) be a topological space. Let $A \subseteq X$. The set A is sequentially compact if A satisfies:

If (a_1, a_2, \dots) is a sequence in A then there exists $z \in A$ such that z is a cluster point of (a_1, a_2, \dots) .

Cauchy compact

In English: Every Cauchy sequence in A has a limit point in A .

In Math: Let (X, d_X) be a metric space. Let $A \subseteq X$. The set A is Cauchy compact, or complete if A satisfies:

if (a_1, a_2, \dots) is a Cauchy sequence in A then there exists $z \in A$ such that z is a limit point of (a_1, a_2, \dots) .

Closed

In English: Every close point to A is in A .

In Math: Let (X, \mathcal{T}_X) be a topological space.

Let $A \subseteq X$. The set A is closed in X if

A satisfies:

$$A^c \in \mathcal{T}_X$$

Examples

(1) Closed \nRightarrow Cauchy compact

Let $X = \mathbb{R}_{(0,1)}$ and let $A = X$ (with the standard metric space topology from $d(x,y) = |y-x|$).

Then A is closed in X and

$(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ is a Cauchy sequence in A which does not converge.

(2) Cauchy compact \nRightarrow sequentially compact.

Let $X = \mathbb{R}$ with the standard metric and let $A = X$. Then A is complete and

$(1, 2, 3, \dots)$ is a sequence in A with no cluster point.

(3) Bounded $\not\Rightarrow$ Ball compact

Let $X = \mathbb{R}$ with metric $d_x: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$

given by

$$d_x(x, y) = \min\{|x - y|, 1\}.$$

Let $A = X$. Then $A \subseteq B_{\frac{1}{2}}(0)$ but A is not covered by a finite number of ε -balls of radius $\frac{1}{10}$.

(4) Ball compact $\not\Rightarrow$ cover compact.

Let $X = \mathbb{R}$ and let $A = \mathbb{R}_{(0, 1)}$ where the metric on \mathbb{R} is the standard metric.

Then A is ball compact but not closed and bounded.

In \mathbb{R}^n (with the standard metric)

A is cover compact \Leftrightarrow A is closed and bounded.