Metric spaces

A <u>metric space</u> is a set X with a function $d_X\colon X\times X\to\mathbb{R}_{\ge 0}$ such that

(a) if $x,y,z\in X$ then
$$d_X(x,y)\le d_X(x,z)+d_X(z,y),$$
(b) If $x,y\in X$ then $d_X(x,y)=d_X(y,x)$,

(c) If $x\in X$ then $d_X(x,x)=0$,

(d) If $x,y\in X$ and $d_X(x,y)=0$ then $x=y$.

<u>ε-ball at a</u> Let $\varepsilon\in\mathbb{R}_{>0}$ and $a\in X$.
$$B_\varepsilon(a)=\{x\in X \mid d_X(x,a)<\varepsilon\}.$$

<u>Neighborhoods of a</u>
$$\mathcal{N}(a)=\left\{N\subseteq X \;\middle|\; \begin{array}{l}\text{there exists } \varepsilon\in\mathbb{R}_{>0}\\ \text{such that } B_\varepsilon(a)\subseteq N\end{array}\right\}$$

<u>Open sets in X</u>: the <u>metric space topology</u>
$$\mathcal{T}_X=\{U\subseteq X \mid \text{if } a\in U \text{ then } U\in\mathcal{N}(a)\}.$$

<u>Closed sets in X</u>
$$\mathcal{C}_X=\{C\subseteq X \mid C^c\in\mathcal{T}_X\},$$
where $C^c=\{x\in X \mid x\notin C\}$.

Topological spaces

A <u>topological space</u> is a set X with a collection $\mathcal{T}_X$ of subsets of X such that

(a) $\emptyset \in \mathcal{T}_X$ and $X \in \mathcal{T}_X$

(b) If $\mathcal{S} \subseteq \mathcal{T}_X$ then $\left(\bigcup_{U \in \mathcal{S}} U\right) \in \mathcal{T}_X$

(c) If $k \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_k \in \mathcal{T}_X$ then
$$U_1 \cap U_2 \cap \cdots \cap U_k \in \mathcal{T}_X.$$

Neighborhoods of a Let $a \in X$.

$$\mathcal{N}(a) = \left\{ N \subseteq X \,\middle|\, \begin{array}{l} \text{there exists } U \in \mathcal{T}_X \text{ with} \\ a \in U \text{ and } U \subseteq N \end{array} \right\}$$

Open sets in X

$$\mathcal{T}_X = \{ U \subseteq X \mid U \in \mathcal{T}_X \}$$

Closed sets in X

$$\mathcal{C}_X = \{ C \subseteq X \mid C^c \in \mathcal{T}_X \}$$

where $C^c = \{ x \in X \mid x \notin C \}$.

Let A be a subset of X, $A \subseteq X$.

An <u>interior point of A</u> is $x \in X$ such that
there exists $N \in \mathcal{N}(x)$ such that $N \subseteq A$.

A <u>close point to A</u> is $x \in X$ such that
if $N \in \mathcal{N}(x)$ then $N \cap A \neq \emptyset$.

The <u>interior</u> of A is the subset A° of X such that

(a) $A^\circ \in \mathcal{T}_X$ and $A^\circ \subseteq A$,

(b) If $U \in \mathcal{T}_X$ and $U \subseteq A$ then $U \subseteq A^\circ$.

(A° is the largest open set contained in A)

The <u>closure</u> of A is the subset $\overline{A}$ of X such that

(a) $\overline{A} \in \mathcal{C}_X$ and $\overline{A} \supseteq A$

(b) If $C \in \mathcal{C}_X$ and $C \supseteq A$ then $C \supseteq \overline{A}$.

($\overline{A}$ is the smallest closed set containing A).

<u>Proposition</u> Let $(X, \mathcal{T}_X)$ be a topological space.
Let $A \subseteq X$.

(a) $A^\circ = \{\text{interior points of } A\}$

(b) $\overline{A} = \{\text{close points to } A\}$.

<u>Proof of (a)</u> Let $I = \{\text{interior points of } A\}$.

To show: ~~(aa)~~ $A^\circ = I$.

To show: (aa) $A^\circ \subseteq I$

(ab) $I \subseteq A^\circ$.

(ab) Let $x \in I$.

Then there exists $N \in \mathcal{N}(x)$ such then $N \subseteq A$.

So there exists $U \in \mathcal{T}_X$ with $x \in U$ and $U \subseteq N$.

Since $U \in \mathcal{T}_X$ and $U \subseteq A$ then $U \subseteq A^\circ$.

So $x \in A^\circ$

So $I \subseteq A^\circ$.

(aa) Assume $x \in A^\circ$.

Since $A^\circ \in \mathcal{T}_X$ and $x \in A^\circ$ and $A^\circ \subseteq A$

then x is an interior point of A.

So $x \in I$.

So $A^\circ \subseteq I$.

So $I = A^\circ$.

Limits and continuity

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

Let $f: X \to Y$ be a function

Let $a \in X$ and $y \in Y$.

$\lim_{x \to a} f(x) = y$ if f satisfies:

if $P \in \mathcal{N}(y)$ then there exists $N \in \mathcal{N}(a)$ such that $P \supseteq f(N)$.

$\lim_{\substack{x \to a \\ x \neq a}} f(x) = y$ if f satisfies:

if $P \in \mathcal{N}(y)$ then there exists $N \in \mathcal{N}(a)$ such that $P \supseteq f(N - \{a\})$

$\lim_{n \to \infty} x_n = z$ if $(x_1, x_2, \ldots)$ satisfies:

if $P \in \mathcal{N}(y)$ then there exists $n \in \mathbb{Z}_{>0}$ such that $P \supseteq \{x_n, x_{n+1}, \ldots\}$.

The function f is continuous at a if f satisfies:

if $V \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.

The function f is continuous if f satisfies:

if $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{T}_X$.