The spectral theorem and eigenvalues

Theorem Let $(H, \langle , \rangle)$ be a Hilbert space. Let $T: H \to H$ be a bounded self adjoint compact linear operator. Then

$$H = \overline{\bigoplus_{\lambda \in \sigma_p(T)} H_\lambda} \quad \text{and}$$

if H is separable then H has a countable orthonormal basis of eigenvectors of T.

One possibility is $H = \mathbb{C}^7$ with

$$\langle (x_1, x_2, \dots, x_7), (y_1, y_2, \dots, y_7) \rangle = x_1 \bar{y}_1 + \cdots + x_7 \bar{y}_7$$

and $T: H \to H$ is the linear transformation which, in the basis $\{e_1, e_2, \dots, e_7\}$, is given by

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 6 & 7 & 8 & 9 \\ 3 & 7 & 10 & 11 & 12 \\ 4 & 8 & 11 & 13 & 14 \\ 5 & 9 & 12 & 14 & 15 \end{pmatrix}$$

where $e_1 = (1, 0, 0, \dots 0)$, $e_2 = (0, 1, 0, \dots 0)$, $\dots$ $e_7 = (0, 0, \dots 0, 1)$.

In terms of matrices

T is self adjoint if $A = \bar{A}^t$. (conjugate transpose).

Then our theorems tell us

$$\|T\| = \sup\left\{ \frac{\|Tu\|}{\|u\|} \,\Big|\, u \in \mathbb{C}^7, \|u\| = 1 \right\}$$

$$= \sup\left\{ |\langle Tu, u\rangle| \,\big|\, u \in \mathbb{C}^7, \|u\| = 1 \right\}$$

$= |\lambda_1|$, where λ_1 is the largest eigenvalue of A.

The basis $\{e_1, \dots, e_q\}$ is orthonormal and the spectral theorem says that there is a orthonormal basis $\{u_1, \dots, u_q\}$ of $\mathbb{C}^q$ consisting of eigenvectors of A.

If U is the change of ~~basis~~ basis matrix from $\{e_1, \dots, e_q\}$ to $\{u_1, \dots, u_q\}$ then

$$U = \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_q \\ | & | & & | \end{pmatrix}$$

(HW: Show that U is unitary, U is an isometry)

Then

$$D = U^{-1} A U = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_q \end{pmatrix}$$

where $\lambda_1, \lambda_2, \dots \lambda_q$ are the eigenvalues of A.

Then

$$\frac{1}{\lambda_1} D = \begin{pmatrix} 1 & & & \\ & \lambda_2/\lambda_1 & & \\ & & \ddots & \\ & & & \lambda_q/\lambda_1 \end{pmatrix} \text{ and } \left(\frac{1}{\lambda_1} D\right)^{1000} = \begin{pmatrix} 1^{1000} & & & 0 \\ & (\lambda_2/\lambda_1)^{1000} & & \\ & & \ddots & \\ 0 & & & (\lambda_q/\lambda_1)^{1000} \end{pmatrix}$$

So

$$\lim_{n \to \infty} \left(\frac{1}{\lambda_1} D\right)^n = \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots \\ & & & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is an eigenvector of D eigenvalue λ_1

So

$$\lim_{n \to \infty} \left(\frac{1}{\lambda_1} A\right)^n = \lim_{n \to \infty} \left(\frac{1}{\lambda_1} U D U^{-1}\right)^n = \lim_{n \to \infty} U \left(\frac{1}{\lambda_1} D\right)^n U^{-1}$$

$$= U \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots \\ & & & 0 \end{pmatrix} U^{-1} \text{ and } \left(\lim_{n \to \infty} \left(\frac{1}{\lambda_1} A\right)^n\right) x$$

is an eigenvector of A of eigenvalue λ_1

Let $B \in M_q(\mathbb{C})$.

Let

$$A = B^*B, \text{ where } B^* = \bar{B}^t.$$

Then

$$\bar{A}^t = \overline{(B^*B)}^t = \bar{B}^t\,\overline{(\bar{B}^t)}^t = B^*B = A.$$

So A is self adjoint.

Let U be unitary such that

$$D = U^{-1}AU = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_q \end{pmatrix}$$ where $\lambda_1, \lambda_2, \dots \lambda_q$ are the eigenvalues of A.

Since D and A have the same eigenvalues (and are both self adjoint and compact) then

$$\|D\| = \|A\| = |\lambda_1| \quad \text{(HW: Prove this directly)}$$

<u>Claim</u> $\|B\| = \max\{\sqrt{|\lambda_1|}, \dots \sqrt{|\lambda_q|}\}$.

(a) If $x \in H$ and $Ax = \lambda_j x$ then

$$\|Bu_j\|^2 = \langle Bu_j, Bu_j\rangle = \langle u_j, B^*Bu_j\rangle$$
$$= \langle u_j, Au_j\rangle = \lambda_j\langle u_j, u_j\rangle = \lambda_j\|u_j\|^2$$

So $\|B\| \geq \sqrt{|\lambda_j|}$ and $\|B\| \geq \max\{\sqrt{|\lambda_1|}, \dots \sqrt{|\lambda_q|}\}$.

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(b) If $x \in H$ and $x = x_1 u_1 + \cdots + x_q u_q \neq 0$ ~~$x = (x_1, \ldots, x_q)$~~ then

$$\|Bx\|^2 = \langle Bx, Bx \rangle = \langle x, B^* Bx \rangle = \langle x, Ax \rangle$$

$$= \langle x_1 u_1 + \cdots + x_q u_q, A(x_1 u_1 + \cdots + x_q u_q) \rangle$$

$$= \langle x_1 u_1 + \cdots + x_q u_q, x_1 \lambda_1 u_1 + \cdots + x_q \lambda_q u_q \rangle$$

$$= x_1^2 \lambda_1 + \cdots + x_q^2 \lambda_q \leq \max\{|\lambda_1|, \ldots, |\lambda_q|\} \|x\|^2$$

So $$\frac{\|Bx\|}{\|x\|} \leq \max\{\sqrt{|\lambda_1|}, \ldots, \sqrt{|\lambda_q|}\}.$$

So $$\|B\| = \max\{\sqrt{|\lambda_1|}, \ldots, \sqrt{|\lambda_q|}\}.$$

This determines $\|B\|$ for any $B \in M_q(\mathbb{C})$.