Constructing eigenvalues and eigenvectors

Let $(H, \langle\ \rangle)$ be a Hilbert space.
Let $T: H \to H$ be a bounded linear operator so that

$$\|T\| = \sup\{\|Tu\| \mid u \in H, \|u\| = 1\}$$

exists in $\mathbb{R}_{\geq 0}$.

The operator $T: H \to H$ is <u>compact</u> if T satisfies:

if $(u_1, u_2, \ldots)$ is a sequence in $\{u \in H \mid \|u\| = 1\}$
then $(Tu_1, Tu_2, \ldots)$ has a cluster point in H.

The operator $T: H \to H$ is <u>self-adjoint</u> if T satisfies:

if $x, y \in H$ then $\langle Tx, y\rangle = \langle x, Ty\rangle$

<u>Theorem 1</u> Let $(H, \langle\ \rangle)$ be a Hilbert space and let $T: H \to H$ be a bounded self adjoint operator. Let

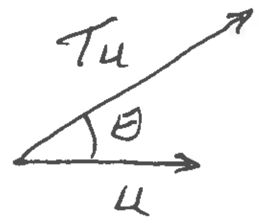
$$\lambda = \sup\{|\langle Tu, u\rangle| \mid u \in H, \|u\| = 1\}$$

Then

$$\lambda = \|T\| \text{ and } \lambda - T \text{ is not a bijection.}$$

<u>Remark</u> By Cauchy-Schwarz,

$$|\langle Tu, u\rangle| \leq \|Tu\| \cdot \|u\|$$



Let $$\theta = \arccos\left(\frac{\langle Tu, u\rangle}{\|Tu\|\,\|u\|}\right)$$

If $\theta = 0$ or $\theta = \pi$ then u is an eigenvector of T

<u>Theorem 2</u> Let $(H, \langle\ \rangle)$ be a Hilbert space and let $T: H \to H$ be a bounded compact self adjoint operator. Let $(u_1, u_2, \ldots)$ be a sequence in $\{u \in H \mid \|u\| = 1\}$ such that

$$\lim_{n \to \infty} |\langle Tu, u\rangle| = \|T\|.$$

Let y be a cluster point of $(Tu_1, Tu_2, \ldots)$

Then

$$\|y\| = \|T\|, \qquad \frac{|\langle Ty, y\rangle|}{\|y\|^2} = \|T\|$$

and $Ty = \|T\| y$

<u>Theorem 3</u> Let $(H, \langle\ \rangle)$ be a Hilbert space and let $T: H \to H$ be a bounded compact self adjoint operator.

Let $b_0 \in H$ and let

$$(b_1, b_2, \ldots) \quad \text{and} \quad (\mu_1, \mu_2, \ldots)$$

be the sequences defined by

$$b_{k+1} = \frac{Tb_k}{\|Tb_k\|} = \frac{T^{k+1}b_0}{\|T^{k+1}b_0\|} \quad \text{and}$$

$$\mu_{k+1} = \frac{\langle Tb_k, b_k\rangle}{\langle b_k, b_k\rangle} = \frac{\langle b_{k+1}, b_k\rangle}{\|b_k\|^2}.$$

Then

$$\|T\| = \lim_{k\to\infty} \mu_k \quad \text{and} \quad v = \lim_{k\to\infty} b_k$$

is an eigenvector of T of eigenvalue $\|T\|$

Theorem 4 Let $(H, \langle , \rangle)$ be a Hilbert space and let $T: H \to H$ be a bounded self adjoint compact operator. For $\lambda \in \mathbb{C}$ let

$$H_\lambda = \{v \in H \mid Tv = \lambda v\} \text{ and let } \sigma_p(H) = \{\lambda \in \mathbb{K} \mid H_\lambda \neq 0\}.$$

(a) If $H_\lambda \neq 0$ then $\lambda \in \mathbb{R}$

(b) If $\lambda \neq \gamma$ then $H_\lambda \perp H_\gamma$

(c) If T is compact and $\lambda \neq 0$ then H_λ is finite dimensional.

(d) If T is compact and $\lambda_1, \lambda_2, \dots$ is a sequence of distinct eigenvalues in $\sigma_p(T)$ then

$$\lim_{k \to \infty} \lambda_k = 0$$

(e) If T is compact then $H = \overline{\left(\bigoplus_{\lambda \in \sigma_p(H)} H_\lambda\right)}$