Eigenvectors and eigenvalues

Theorem Let H be a $\mathbb{C}$-vector space.
Let $T: H \to H$ be a linear transformation.

(a) T has an eigenvector of eigenvalue λ
if and only if
$T-\lambda$ is not injective

(b) Assume T is a compact bounded operator.
The following are equivalent.
(i) $T-\lambda$ is not injective
(ii) $T-\lambda$ is not bijective
(iii) $\det(T-\lambda)=0$.

Definitions

Let H be a $\mathbb{C}$-vector space
Let $T: H \to H$ be a linear transformation.
Let $\lambda \in \mathbb{C}$. The <u>λ-eigenspace of T</u> is

$$H_\lambda = \{v \in H \mid Tv = \lambda v\}.$$

An <u>eigenvector of T of eigenvalue λ</u> is
$v \in H_\lambda$ such that $v \neq 0$.

The <u>point spectrum of T</u> is

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid H_\lambda \neq 0\}.$$

Bounded and compact operators

Let $(H, \| \ \|)$ be a normed $\mathbb{C}$-vector space and let $T: H \to H$ be a linear operator.

The operator T is <u>bounded</u> if T satisfies:

$\|T\|$ exists in $\mathbb{R}_{\geq 0}$, where

$$\|T\| = \sup\left\{ \frac{\|Tv\|}{\|v\|} \mid v \in H, v \neq 0 \right\}$$

$$= \sup\{ \|Tu\| \mid u \in H, \|u\| = 1 \}.$$

The operator T is <u>compact</u> if T satisfies

if $(x_1, x_2, \ldots)$ is a sequence in $\{u \in H \mid \|u\| \leq 1\}$ then $(Tx_1, Tx_2, \ldots)$ has a cluster point in H.

Let (X, d) be a metric space.

Let $(x_1, x_2, \ldots)$ be a sequence in X.

A <u>cluster point</u> of $(x_1, x_2, \ldots)$ in X is $z \in X$ such that

there exists a subsequence $(x_{n_1}, x_{n_2}, \ldots)$ of $(x_1, x_2, \ldots)$ such that $\lim_{k \to \infty} x_{n_k} = z$

<u>Example</u> The sequence $(a_1, a_2, \ldots)$ in $\mathbb{R}$ given by $a_n = (-1)^{n-1}(1 + \frac{1}{n})$ has cluster points at 1 and -1 but no limit point.

<u>Example 1</u> Let $H = \mathbb{C}[x]$ and $T = \frac{d}{dx}$.

The matrix of T with respect to the basis $\{1, x, x^2, \dots\}$ is

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ & & 0 & 1 & \\ 0 & & & 0 & \ddots \end{pmatrix}$$

Let $\lambda \in \mathbb{C}$.

<u>Case 1: $\lambda = 0$</u>: The eigenvectors of eigenvalue 0 are the constant polynomials,

$$H_0 = \left\{ a + 0x + 0x^2 + \dots + 0x^k \;\middle|\; \begin{matrix} k \in \mathbb{Z}_{\geq 0} \text{ and} \\ a_i \in \mathbb{C} \end{matrix} \right\}$$

and $\dim(H_0) = 1$.

<u>Case 2: $\lambda \neq 0$</u>: T has no eigenvector of eigenvalue λ, i.e.

if $\lambda \neq 0$ then $H_\lambda = \{0\}$

<u>Example 2</u> Let $H = \{ f : \mathbb{C} \to \mathbb{C} \mid f \text{ is differentiable} \}$ and let $T = \frac{d}{dx}$.

Let $\lambda \in \mathbb{C}$. Then $v = e^{\lambda x}$ is an eigenvector of eigenvalue λ, i.e.

if $\lambda \in \mathbb{C}$ then $H_\lambda \neq 0$.

<u>HW</u> Show that $B = \left\{ \frac{1}{x - \lambda} \;\middle|\; \lambda \in \mathbb{C} \right\}$ is linearly independent in $\mathbb{C}[[x]]$.

<u>Example 3</u> Let $H = \mathbb{C}[x]$ and let $T = x$.

The matrix of T with respect to the basis $\{1, x, x^2, \ldots\}$ is

$$B = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & \ddots \end{pmatrix}$$

Let $\lambda \in \mathbb{C}$. T does <u>not</u> have an eigenvector of eigenvalue λ, i.e.

$$H_\lambda = \{0\} \quad \text{for } \lambda \in \mathbb{C}.$$

Note that

$$AB = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & \ddots \end{pmatrix} \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & 0 \\ & & & \ddots \end{pmatrix} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \end{pmatrix}$$

So A is "invertible".

<u>Example</u> $H = \mathbb{C}[x]$ and $T = \frac{d}{dx}$.

$H = \mathbb{C}[[x]]$ and $T = \frac{d}{dx}$.

In the basis $\{1, x, x^2, \ldots\}$, T has matrix

$$T = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & \ddots \end{pmatrix}$$