5 Completions

The point of this chapter is to introduce Cauchy filters, Cauchy sequences, complete spaces and completions.

Theorem 5.1.

(a) Let $(X, \mathcal{X})$ be a uniform space. There exists a unique completion $(\widehat{X}, \widehat{\mathcal{X}}, \iota\colon X \to \widehat{X})$ of X.

(b) Let (X, d) be a metric space. There exists a unique completion $(\widehat{X}, \hat{d}, \iota\colon X \to \widehat{X})$ of X.

5.1 Cauchy sequences and complete metric spaces

Let (X, d) be a metric space. A sequence $(x_1, x_2, \ldots)$ in X *converges* if there exists $z \in X$ such that

$$\text{if } \varepsilon \in \mathbb{R}_{>0} \text{ then there exists } \ell \in \mathbb{Z}_{>0} \text{ such that if } n \in \mathbb{Z}_{\geq \ell} \text{ then } d(x_n, z) < \varepsilon.$$

A *Cauchy sequence* in X is a sequence $(x_1, x_2, \ldots)$ in X such that

$$\text{if } \varepsilon \in \mathbb{R}_{>0} \quad \text{then there exists } \ell \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } m, n \in \mathbb{Z}_{\geq \ell} \text{ then } d(x_m, x_n) < \varepsilon.$$

A metric space (X, d) is *complete*, or *Cauchy compact*, if every Cauchy sequence in X converges.

5.1.1 Completion of a metric space

Let (X, d_X) and (Y, d_Y) be metric spaces. An *isometry from X to Y* is a function $\varphi\colon X \to Y$ such that

$$\text{if } x_1, x_2 \in X \quad \text{then} \quad d_Y(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2).$$

Let (X, d) be a metric space. The *completion of* (X, d) is a metric space $(\widehat{X}, \hat{d})$ with an isometry

$$\iota\colon X \to \widehat{X} \quad \text{such that} \qquad (\widehat{X}, \hat{d}) \text{ is complete} \qquad \text{and} \qquad \overline{\iota(X)} = \widehat{X},$$

where $\overline{\iota(X)}$ is the closure of the image of ι.

5.1.2 Existence of the completion of a metric space

Let (X, d) be a metric space. The *completion* of X is the metric space

$$\widehat{X} = \{\text{Cauchy sequences } \vec{x} \text{ in } X\} \qquad \text{with the function} \qquad \begin{matrix} \iota\colon & X & \longrightarrow & \widehat{X} \\ & x & \longmapsto & (x, x, x, \ldots) \end{matrix}$$

where $\widehat{X}$ has the metric

$$d\colon \widehat{X} \times \widehat{X} \to \mathbb{R}_{\geq 0} \qquad \text{defined by} \qquad d(\vec{x}, \vec{y}) = \lim_{n \to \infty} d(x_n, y_n),$$

and *Cauchy sequences $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$ are equal in $\widehat{X}$*,

$$\vec{x} = \vec{y} \qquad \text{if } \lim_{n \to \infty} d(x_n, y_n) = 0.$$

5.1.3 Cauchy filters and complete uniform spaces

Let $(X, \mathcal{X})$ be a uniform space.

Let $E \in \mathcal{X}$ and $x \in X$. The *E-neighborhood of x* is

$$B_E(x) = \{y \in X \mid (x, y) \in E\}.$$

Let $x \in X$. The *neighborhood filter* of x is

$$\mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } E \in \mathcal{X} \text{ such that } N \supseteq B_E(x).\}$$

A filter $\mathcal{F}$ on X *converges* if there exists $z \in X$ such that $\mathcal{F} \supseteq \mathcal{N}(z)$.
A sequence $(x_1, x_2, \ldots)$ in X *converges* if there exists $z \in X$ such that

$$\text{if } N \in \mathcal{N}(z) \text{ then there exists } \ell \in \mathbb{Z}_{>0} \text{ such that if } n \in \mathbb{Z}_{\geq \ell} \text{ then } x_n \in N.$$

A *Cauchy filter* is a filter $\mathcal{F}$ on X such that

$$\text{if } E \in \mathcal{X} \text{ then there exists } N \in \mathcal{F} \text{ such that } N \times N \subseteq E.$$

A *Cauchy sequence* is a sequence $\vec{x} = (x_1, x_2, \ldots)$ in X such that

$$\text{if } E \in \mathcal{X} \text{ then there exists } \ell \in \mathbb{Z}_{>0} \text{ such that} \qquad \text{if } m, n \in \mathbb{Z}_{\geq \ell} \text{ then } (x_m, x_n) \in E.$$

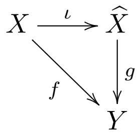
A *complete space* is a uniform space for which every Cauchy filter on X converges.

5.1.4 Completion of a uniform space

Let $(X, \mathcal{X})$ be a uniform space. A *completion of* X is a complete Hausdorff uniform space $(\widehat{X}, \widehat{\mathcal{X}})$ with a uniformly continuous function $\iota\colon X \to \widehat{X}$ such that

if Y is a complete Hausdorff uniform space and $f\colon X \to Y$ is a uniformly continuous map

then there exists a unique uniformly continuous function $g\colon \widehat{X} \to Y$ such that $f = g \circ \iota$.



5.1.5 Existence of the completion of a uniform space

Let $(X, \mathcal{X})$ be a uniform space. A *minimal Cauchy filter* on X is a Cauchy filter which is minimal with respect to inclusion of filters. An element

$$V \in \mathcal{X} \text{ is } \textit{symmetric} \text{ if } V \text{ satisfies:} \qquad \text{if } (x, y) \in V \text{ then } (y, x) \in V.$$

For $x \in X$, let $\mathcal{N}(x)$ be the neighborhood filter of x.

The *completion* of X is the uniform space

$$\widehat{X} = \{\text{minimal Cauchy filters } \hat{x} \text{ on } X\} \qquad \text{with the function} \quad \begin{array}{rccc} \iota\colon & X & \longrightarrow & \widehat{X} \\ & x & \longmapsto & \mathcal{N}(x) \end{array}$$

with the uniformity

$$\widehat{\mathcal{X}} = \{U \subseteq \widehat{X} \times \widehat{X} \mid U \text{ contains } \hat{V} \text{ for a symmetric } V \in \mathcal{X}\},$$

where

$$\hat{V} = \{(\hat{x}, \hat{y}) \in \widehat{X} \times \widehat{X} \mid \text{there exists } N \in \hat{x} \cap \hat{y} \text{ such that } N \times N \subseteq V\}.$$

5.2 Notes and references

The treatment of metric spaces and completion follows [BR] Chapter 2 Exercise 24.

The basic material on completions given in §1 can be found in many books, in particular, [AMa1969] Chapt 10. The p-adic integers $\mathbb{Z}_p$ and the p-adic numbers $\mathbb{Q}_p$ are treated in [Bou, Top. Ch. III §6 Ex. 23 and 24 and §7 Ex. 1].

5.3 Some proofs

5.3.1 Construction of the completion of a metric space

Theorem 5.2. *Let (X, d) be a metric space. Let $(\widehat{X}, \hat{d}, \varphi)$ be the metric space*

$$\widehat{X} = \{\text{Cauchy sequences } \vec{x} \text{ in } X\} \qquad \text{with the function} \quad \begin{array}{rccc} \varphi\colon & X & \longrightarrow & \widehat{X} \\ & x & \longmapsto & (x, x, x, \ldots) \end{array}$$

where $\widehat{X}$ has the metric

$$\hat{d}\colon \widehat{X} \times \widehat{X} \to \mathbb{R}_{\geq 0} \qquad \text{defined by} \qquad \hat{d}(\vec{x}, \vec{y}) = \lim_{n\to\infty} d(x_n, y_n),$$

and Cauchy sequences $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$ are equal in $\widehat{X}$,

$$\vec{x} = \vec{y} \qquad \text{if } \lim_{n\to\infty} d(x_n, y_n) = 0.$$

Then $(\widehat{X}, \hat{d})$ with the isometry $\iota\colon X \to \widehat{X}$ such that

$$(\widehat{X}, \hat{d}) \text{ is a complete metric space} \qquad \text{and} \qquad \overline{\varphi(X)} = \widehat{X},$$

where $\overline{\varphi(X)}$ is the closure of the image of φ.

Proof.
To show: (a) $(\widehat{X}, \hat{d})$ is a metric space.
(b) $(\widehat{X}, \hat{d})$ is complete.
(c) $\varphi\colon X \to \widehat{X}$ is an isometry.
(d) $\overline{\varphi(X)} = \widehat{X}$.

(c) To show: If $x, y \in X$ then $\hat{d}(\varphi(x), \varphi(y)) = d(x, y)$.
Assume $x, y \in X$.

$$\hat{d}(\varphi(x), \varphi(y)) = \lim_{n to \infty} d(\varphi(x)_n, \varphi(y)_n) = \lim_{n\to\infty} d(x, y) = d(x, y).$$

So φ is an isometry.

(a) To show: $(\widehat{X}, \hat{d})$ is a metric space.
To show: (aa) $\hat{d}\colon \widehat{X} \times \widehat{X} \to \mathbb{R}_{\geq 0}$ given by $\hat{d}(\vec{x}, \vec{y}) = \lim_{n\to\infty} d(x_n, y_n)$ is a function.
(ab) If $\vec{x}, \vec{y} \in \widehat{X}$ then $\hat{d}(\vec{x}, \vec{y}) = \hat{d}(\vec{y}, \vec{x})$.
(ac) If $\vec{x} \in \widehat{X}$ then $\hat{d}(\vec{x}, \vec{x}) = 0$.
(ad) If $\vec{x}, \vec{y} \in \widehat{X}$ and $\hat{d}(\vec{x}, \vec{y}) = 0$ then $\vec{x} = \vec{y}$.
(ab) If $\vec{x}, \vec{y}, \vec{z} \in \widehat{X}$ then $\hat{d}(\vec{x}, \vec{y}) \leq \hat{d}(\vec{x}, \vec{y}) + \hat{d}(\vec{y}, \vec{z})$.

(aa) To show: If $\vec{x}, \vec{y} \in \widehat{X}$ then there exists a unique $z \in \mathbb{R}_{\geq 0}$ such that $z = \lim_{n\to\infty} d(x_n, y_n)$.
Assume $\vec{x}, \vec{y} \in \widehat{X}$ with $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$.
Let $d_1, d_2, \ldots$ be the sequence in $\mathbb{R}_{\geq 0}$ given by

$$d_n = d(x_n, y_n).$$

To show: There exists $z \in \mathbb{R}_{\geq 0}$ such that $z = \lim_{n\to\infty} d_n$.

Since $\mathbb{R}_{\ge 0}$ is a metric space, and metric spaces are Hausdorff, HERE WE USE THAT METRIC SPACES ARE HAUSDORFFand limits in Hausdorff spaces are unique when they exist, the limit z will be unique if it exists.
To show: $d_1, d_2, \ldots$ is a Cauchy sequence in $\mathbb{R}_{\ge 0}$. This will show that z exists since $\mathbb{R}_{\ge 0}$ is complete HERE WE USE THAT $\mathbb{R}_{>0}$ IS A COMPLETE METRIC SPACE and Cauchy sequences in complete spaces converge.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\ge N}$ then $|d_m - d_n| < \epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
Let $N = \max(N_1, N_2)$, where

N_1 is such that if $n, m \in \mathbb{Z}_{\ge N_1}$ then $d(x_m, x_n) \in \frac{\epsilon}{2}$, and
N_2 is such that if $n, m \in \mathbb{Z}_{\ge N_2}$ then $d(y_m, y_n) \in \frac{\epsilon}{2}$.

(N_1 and N_2 exist since $\vec{x}$ and $\vec{y}$ are Cauchy sequences.)
Assume $m, n \in \mathbb{Z}_{\ge N}$.
To show: $|d_m - d_n| < \epsilon$.

$$|d_m - d_n| = \big|d(x_m, y_m) - d(x_n, y_n)\big| \le \big|d(x_n, x_m) + d(y_n, y_m)\big|,$$

since $d(x_n, y_n) \le d(x_n, x_m) + d(x_n, y_n) + d(y_n, y_m)$.
So

$$|d_m - d_n| \le \big|d(x_n, x_m) + d(y_n, y_m)\big| \le \big|d(x_n, x_m)\big| + \big|d(y_n, y_m)\big| < \epsilon_2 + \epsilon_2 = \epsilon.$$

So $d_1, d_2, \ldots$ is a Cauchy sequence in $\mathbb{R}_{\ge 0}$.
So $z = \lim_{n\to\infty} d_n$ exists in $\mathbb{R}_{\ge 0}$.

(ab) To show: If $\vec{x}, \vec{y} \in \widehat{X}$ then $\hat{d}(\vec{x}, \vec{y}) = \hat{d}(\vec{y}, \vec{x})$.
Assume $\vec{x}, \vec{y} \in \widehat{X}$ with $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$.
Since $d(x_n, y_n) = d(y_n, x_n)$,

$$\hat{d}(\vec{x}, \vec{y}) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = \hat{d}(\vec{y}, \vec{x}).$$

(ac) To show: If $\vec{x} \in \widehat{X}$ then $\hat{d}(\vec{x}, \vec{x}) = 0$.
Assume $\vec{x} \in \widehat{X}$.
To show $\hat{d}(\vec{x}, \vec{x}) = 0$.
Since $d(x_n, x_n) = 0$,

$$\hat{d}(\vec{x}, \vec{x}) = \lim_{n\to\infty} d(x_n, x_n) = \lim_{n\to\infty} 0 = 0.$$

(ad) If $\vec{x}, \vec{y}, \vec{z} \in \widehat{X}$ then $\hat{d}(\vec{x}, \vec{y}) \le \hat{d}(\vec{x}, \vec{z}) + \hat{d}(\vec{z}, \vec{y})$.
Assume $\vec{x}, \vec{y}, \vec{z} \in \widehat{X}$.
To show: $\hat{d}(\vec{x}, \vec{y}) \le \hat{d}(\vec{x}, \vec{z}) + \hat{d}(\vec{z}, \vec{y})$.

$$\begin{aligned}\hat{d}(\vec{x}, \vec{y}) &= \lim_{n\to\infty} d(x_n, y_n) \le \lim_{n\to\infty} \big(d(x_n, z_n) + d(z_n, y_n)\big) \\ &= \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n) = \hat{d}(\vec{x}, \vec{z}) + \hat{d}(\vec{z}, \vec{y}),\end{aligned}$$

where the next to last equality follows from the continuity of addition in $\mathbb{R}_{\ge 0}$.

(d) To show: $\overline{\varphi(X)} = \widehat{X}$.
To show: If $\vec{z} \in \widehat{X}$ then there exists a sequence $\vec{x}_1, \vec{x}_2, \ldots$ in $\varphi(X)$ such that $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.
Assume $\vec{z} = (z_1, z_2, \ldots) \in \widehat{X}$.

To show: There exists $\vec{x}_1, \vec{x}_2, \ldots$ in $\varphi(X)$ with $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.
Let

$$\begin{aligned} \vec{x}_1 &= (z_1, z_1, z_1, z_1, \ldots) = \varphi(z_1), \\ \vec{x}_2 &= (z_1, z_1, z_1, z_1, \ldots) = \varphi(z_1), \\ \vec{x}_3 &= (z_1, z_1, z_1, z_1, \ldots) = \varphi(z_1), \qquad \ldots \end{aligned}$$

so that $\vec{x}_1, \vec{x}_2, \ldots$ is the sequence $\varphi(z_1), \varphi(z_2), \ldots$ in $\varphi(X)$.
To show: $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.
To show: $\lim_{n\to\infty} \hat{d}(\vec{x}_n, \vec{z}) = 0$.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $n \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $\hat{d}(\vec{x}_n, \vec{z}) < \epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
Let $N \in \mathbb{Z}_{>0}$ be such that if $r, s \in \mathbb{Z}_{\geq N}$ then $d(z_r, z_s) < \epsilon/2$.
The value N exists since $\vec{z} = (z_1, z_2, \ldots)$ is a Cauchy sequence in X.
To show: If $n \in \mathbb{Z}_{\geq N}$ then $\hat{d}(\vec{x}_n, \vec{z}) < \epsilon$.
Assume $n \in \mathbb{Z}_{\geq N}$.
To show: $\hat{d}(\vec{x}_n, \vec{z}) < \epsilon$.
To show: $\lim\limits_{k\to\infty} d((\vec{x}_n)_k, z_k) < \epsilon$.

$$\lim_{k\to\infty} d((\vec{x}_n)_k, z_k) = \lim_{k\to\infty} d(z_n, z_k) \leq \frac{\epsilon}{2} < \epsilon, \qquad \text{since } d(z_n, z_k) < \tfrac{\epsilon}{2} \text{ for } k > N.$$

So $\lim\limits_{n\to\infty} \vec{x}_n = \vec{z}$.
So $\overline{\varphi(X)} = \widehat{X}$.

(b) To show: $(\widehat{X}, \hat{d})$ is complete.
To show: If $\vec{x}_1, \vec{x}_2, \ldots$ is a Cauchy sequence in $\widehat{X}$ then $\vec{x}_1, \vec{x}_2, \ldots$ converges.
Assume

$$\begin{aligned} \vec{x}_1 &= (x_{11}, x_{12}, x_{13}, \ldots), \\ \vec{x}_2 &= (x_{21}, x_{22}, x_{23}, \ldots), \\ \vec{x}_3 &= (x_{31}, x_{32}, x_{33}, \ldots), \\ &\vdots \end{aligned}$$

is a Cauchy sequence in $\widehat{X}$.
To show: There exists $\vec{z} = (z_1, z_2, \ldots)$ in $\widehat{X}$ such that $\lim\limits_{n\to\infty} \vec{x}_n = \vec{z}$.
Using that $\overline{\varphi(X)} = \widehat{X}$, for $k \in \mathbb{Z}_{>0}$ let $z_k \in X$ be such that $\hat{d}(\varphi(z_k), \vec{x}_k) < \frac{1}{k}$.

$$\begin{array}{lll} \vec{x}_1 = (x_{11}, x_{12}, x_{13}, \ldots), & \varphi(z_1) = (z_1, z_1, z_1, z_1, \ldots), & \hat{d}(\varphi(z_1), \vec{x}_1) < 1, \\ \vec{x}_2 = (x_{21}, x_{22}, x_{23}, \ldots), & \varphi(z_2) = (z_2, z_2, z_2, z_2, \ldots), & \hat{d}(\varphi(z_2), \vec{x}_2) < \frac{1}{2}, \\ \vec{x}_3 = (x_{31}, x_{32}, x_{33}, \ldots), & \varphi(z_3) = (z_3, z_3, z_3, z_3, \ldots), & \hat{d}(\varphi(z_3), \vec{x}_3) < \frac{1}{3}, \\ \quad\vdots & \quad\vdots & \quad\vdots \end{array}$$

To show: (ba) $\vec{z} = (z_1, z_2, z_3, \ldots)$ is a Cauchy sequence.

(bb) $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.

(ba) To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(z_r, z_s) < \epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(z_r, z_s) < \epsilon$.
Let $\ell_1 = \left\lceil \frac{3}{\epsilon} \right\rceil + 1$, so that $\frac{1}{\ell_1} < \frac{\epsilon}{3}$.
Let $\ell_2 \in \mathbb{Z}_{>0}$ be such that if $r, s \in \mathbb{Z}_{\geq \ell_2}$ then $\hat{d}(\vec{x}_r, \vec{x}_s) < \frac{\epsilon}{3}$.
Let $\ell = \max\{\ell_1, \ell_2\}$.
To show: If $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(z_r, z_s) < \epsilon$.
Assume $r, s \in \mathbb{Z}_{\geq \ell}$.
To show: $d(z_r, z_s) < \epsilon$.

$$\begin{aligned} d(z_r, z_s) &= \hat{d}(\varphi(z_r), \varphi(z_s)) \leq \hat{d}(\varphi(z_r), \vec{x}_r) + \hat{d}(\vec{x}_r, \vec{x}_s) + \hat{d}(\vec{x}_s, \varphi(z_s)) \\ &\leq \frac{1}{r} + \frac{\epsilon}{3} + \frac{1}{s} < \frac{1}{\ell_1} + \frac{\epsilon}{3} + \frac{1}{\ell_1} = \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

So $\vec{z}$ is a Cauchy sequence.

(bb) To show $\lim_{n\to\infty} \hat{d}(\vec{x}_n, \vec{z}) = 0$.

$$\begin{aligned} \lim_{n\to\infty} \hat{d}(\vec{x}_n, \vec{z}) &\leq \lim_{n\to\infty} \big(\hat{d}(\vec{x}_n, \varphi(z_n)) + \hat{d}(\varphi(z_n), \vec{z})\big) \leq \lim_{n\to\infty} \big(\tfrac{1}{n} + \hat{d}(\varphi(z_n), \vec{z})\big) \\ &= \lim_{n\to\infty} \tfrac{1}{n} + \lim_{n\to\infty} \hat{d}(\varphi(z_n), \vec{z}) = 0 + 0 = 0. \end{aligned}$$

So $(\widehat{X}, \hat{d})$ is complete.

So $(\widehat{X}, \hat{d})$ with $\varphi\colon X \to \widehat{X}$ is a completion of X.

□

5.3.2 Construction of the completion of a uniform space

A *minimal Cauchy filter* is a Cauchy filter $\mathcal{F}$ such that if $\mathcal{G}$ is a Cauchy filter and $\mathcal{G} \subseteq \mathcal{F}$ then $\mathcal{G} = \mathcal{F}$.
If $\mathcal{F}$ is a Cauchy filter on X then

$$\mathcal{G} = \{N \subseteq X \mid \text{there exists } E \in \mathcal{E} \text{ and } L \in \mathcal{F} \text{ such that } \sigma(E) = E \text{ and } N \supseteq B_E(L)\}$$

is a minimal Cauchy filter such that $\mathcal{G} \subseteq \mathcal{F}$.

Theorem 5.3. *Let $(X, \mathcal{X})$ be a uniform space. Let $(\widehat{X}, \widehat{\mathcal{X}}, \iota)$ be the uniform space given by the set*

$$\hat{X} = \{\hat{x} \mid \hat{x} \text{ is a minimal Cauchy filter on } X\}$$

with uniformity

$$\hat{\mathcal{X}} = \{\hat{E} \subseteq \hat{X} \times \hat{X} \mid \text{there exists } V \in \mathcal{X} \text{ with } V = \sigma(V) \text{ such that } \hat{E} \supseteq \hat{V}\},$$

where

$$\hat{V} = \{(\hat{x}, \hat{y}) \in \hat{X} \times \hat{X} \mid \text{there exists } M \in \hat{x} \cap \hat{y} \text{ with } M \times M \subseteq V\},$$

and

$$\iota\colon X \to \hat{X} \quad \text{is given by} \quad \iota(x) = \mathcal{N}(x),$$

the neighborhood filter of x in X.

Proof.
To show: (a) $\hat{\mathcal{X}}$ is a uniformity.
(b) $\widehat{X}$ is Hausdorff.
(c) ι is uniformly continuous.
(d) $\overline{\iota(X)} = \widehat{X}$.
(e) $\widehat{X}$ is complete.
(f) $(\widehat{X}, \widehat{\mathcal{X}}, \iota)$ satisfies the universal property.
(a) To show: (aa) If $V \in \mathcal{X}$ then $\Delta(\hat{X}) \subseteq \hat{V}$.
(ab) If $V_1, V_2 \in \mathcal{X}$ then there exists $W \in \mathcal{X}$ such that $\hat{W} \subseteq \hat{V}_1 \cap \hat{V}_2$.
(ac) If $V \in \mathcal{X}$ then there exists $D \in \mathcal{X}$ such that $\hat{D} \subseteq \sigma(\hat{V})$.
(ad) If $V \in \mathcal{X}$ then there exists $W \in \mathcal{X}$ such that $\hat{W} \times_{\hat{X}} \hat{W} \subseteq \hat{V}$.

(aa) Let $V \in \mathcal{X}$ such that $\sigma(V) = V$.
Since $\hat{x} \in \hat{X}$ is a Cauchy filter then $(\hat{x}, \hat{x}) \in \hat{V}$.
(ab) Let $V_1, V_2 \in \mathcal{X}$ such that $\sigma(V_1) = V_1$ and $\sigma(V_2) = V_2$. Then $W = V_1 \cap V_2 \in \mathcal{X}$ and $\sigma(W) = W$.
If $N \subset X$ and $N \times N \subseteq W$ then $N \times N \subseteq V_1$ and $N \times N \subseteq V_2$.
Then $\hat{W} \subseteq \hat{V}_1 \cap \hat{V}_2$.
(ac) By definition of $\hat{V}$, $\sigma(\hat{V}) = \hat{V}$.
(ad) Let $V \in \mathcal{X}$ with $\sigma(V) = V$ and let $W \in \mathcal{X}$ such that $\sigma(W) = W$ and $V \circ V \subseteq W$.??or V??
Let $\hat{x}, \hat{y}, \hat{z} \in \hat{X}$ with $(\hat{x}, \hat{y}) \in \hat{W}$ and $(\hat{y}, \hat{z}) \in \hat{W}$.
Then there exists $M \subseteq X$ and $N \subseteq X$ sucht ath $M \times M \subseteq W$ and $N \times N \subseteq W$ and $M \in \hat{x} \cap \hat{y}$ and $N \in \hat{y} \cap \hat{z}$.
Since $M \in \hat{y}$ and $N \in \hat{y}$ then $M \cap N \neq \emptyset$.
So $(M \cup N) \times (M \cup N) \subseteq W \circ W$
So $(M \cup N) \times (M \cup N) \subseteq V$.
Since $M \cup N \in \hat{x}$ and $M \cup N \in \hat{z}$ then $\hat{W} \circ \hat{W} \subseteq \hat{V}$.

(b) To show: $\widehat{X}$ is Hausdorff.

Let $\hat{x}, \hat{y} \in \hat{X}$ such that there does not exist open sets separating them.
Then $\hat{x}$ and $\hat{y}$ are minimal Cauchy filters in X such that $(\hat{x}, \hat{y}) \in \hat{V}$ for all symmetric $V \in \mathcal{X}$.
Let
$$\hat{z} = \{M \cup N \mid M \in \hat{x} \text{ and } N \in \hat{y}\}_{\subseteq}$$
Then $\hat{z} \subseteq \hat{x}$ and $\hat{z} \subseteq \hat{y}$.
Also $\hat{z}$ is a Cauchy filter (since if $V \in \mathcal{X}$ is symmetric then there exists $P \in \mathcal{X}$ such that $P \times P \in V$, $P \in \hat{x}$ and $P \in \hat{y}$ so that $P \in \hat{z}$).
Since $\hat{x}$ and $\hat{y}$ are minimal Cauchy filters then $\hat{x} = \hat{y} = \hat{z}$.
So $\hat{X}$ is Hausdorff.

(c) To show: ι is uniformly continuous.

Assume $V \in \mathcal{X}$ is symmetric.
Recall that $\hat{V} = \{(\hat{x}, \hat{y}) \in \hat{X} \times \hat{X} \mid \text{there exists } M \in \hat{x} \cap \hat{y} \text{ with } M \times M \subseteq V\}$.
To show: $(i \times i)^{-1}(\hat{V}) \subseteq V \cap (i \times i)^{-1}(\widehat{V \circ V \circ V})$.
If $x, y \in X$ and $(i(x), i(y)) = (i \times i)(x, y) \in \hat{V}\}$ then there exists M such that $M \times M \subseteq V$ and $M \in \mathcal{N}(x)$ and $M \in \mathcal{N}(y)$.

So $(x, y) \in V$.
So $(i \times i)^{-1}(\hat{V}) \subseteq V$.
If $(x, y) \in V$ then $(B_V(x) \cap B_V(y)) \times (B_V(x) \cap B_V(y))) \subseteq V \circ V \circ V$ and $B_V(x) \cap B_V(y) \in \mathcal{N}(x)$ and $B_V(x) \cap B_V(y) \in \mathcal{N}(y)$.

(d) To show: $\overline{\iota(X)} = \widehat{X}$.

Let $\hat{x} \in \widehat{X}$ and let $V \in \mathcal{X}$ be symmetric so that $\hat{V} \in \hat{\mathcal{X}}$.
Let $M = \bigcup_{\substack{E \in \hat{x} \\ E \times E \subseteq V}} E^\circ$.
By (no. 2 Prop. 5 Cor. 4), $M \in \hat{x}$.
Since

$$B_{\hat{V}}(\hat{x}) \cap \iota(X) = \{\iota(x) \mid x \in X \text{ and } (\hat{x}, \iota(x)) \in \hat{V}\}$$

then there exists $x \in X$ and $N \in \mathcal{N}(x)$ such that $N \times N \subseteq V$ with $N \in \hat{x}$.
So there exists $E \subseteq \hat{X}$ with $x \in E$ and $E \subseteq B_{\hat{V}}(\hat{x})$.
So $x \in E^\circ$.
So $B_{\hat{V}}(\hat{x}) \cap \iota(X) = \iota(M)$.
So $B_{\hat{V}}(\hat{x}) \cap \iota(X) \neq \emptyset$.
So $\iota(X)$ is dense in $\widehat{X}$.

(e) To show $\widehat{X}$ is complete.

Let $\mathcal{F}$ be a Cauchy filter on $\iota(X)$.
Since $\iota \colon X \to \hat{X}$ is uniformly continuous then $(\iota^{-1}(\mathcal{F}))_{\subseteq}$ is a Cauchy filter on X.

$$(\iota^{-1}(\mathcal{F}))_{\subseteq} = \{U \subseteq X \mid U \text{ contains a set in } \iota^{-1}(\mathcal{F})\}$$

Let $\hat{x}$ be a minimal Cauchy filter on X with $\hat{x} \subseteq (\iota^{-1}(\mathcal{F}))_{\subseteq}$.
Then $\iota(\hat{x})_{\subseteq}$ is a Cauchy filter on $\iota(X)$.
Also $\mathcal{F} = \iota(\iota^{-1}(\mathcal{F})) \supseteq \iota(\hat{x})_{\subseteq}$.
Since $\overline{\iota(X)} = \hat{X}$ and $\iota(\hat{x})_{\subseteq}$ converges in $\hat{x}$ then $\mathcal{F}$ converges in $\widehat{X}$.
So $\widehat{X}$ is complete.

(f) To show: $(\widehat{X}, \widehat{\mathcal{X}}, \iota)$ satisfies the universal property.

Let Y be a complete Hausdorff uniform space and let $f \colon X \to Y$ be a uniformly continuous function.
Define $g_0 \colon \iota(X) \to Y$ by

$$g_0(\iota(x)) = \lim f(\mathcal{N}(x)).$$

Since f is continuous then $f(x) = \lim f(\mathcal{N}(x)) = g(\iota(x)$.
So $f = g_0 \circ \iota$.
To show: g_0 is uniformly continuous.
Let $U \in \mathcal{X}_Y$ and $V \in \mathcal{X}_X$ with $\sigma(V) = V$ and such that

$$\text{if } (x_1, x_2) \in V \text{ then } (f(x_1), f(x_2)) \in U.$$

Since (by the way that we proved that ι is uniformly continuous??) $\iota\colon X \to \hat{X}$ is uniformly continuous then $(\iota(x_1), \iota(x_2)) \in \hat{V}$ implies $(x_1, x_2) \in V$.
Then $(g_0(\iota(x_1)), g_0(\iota(x_2))) = (f(x_1), f(x_2)) \in U$.
So g_0 is uniformly continuous.
Now, using that $\overline{\iota(X)} = \hat{X}$, let $g\colon \hat{X} \to Y$ be the continuous extension of $g_0\colon \iota(X) \to Y$.
Then $g\colon \hat{X} \to Y$ is the universal property map that we need.

□