4 Existence and uniqueness of limits

The point of this section is to determine conditions on a space for limits to exist (comapactness) and for limits to be unique (Hausdorff spaces).

4.1 Compactness in metric spaces

A *strict metric space* is a set X with a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$ such that

(a) (diagonal condition) If $x \in X$ then $d(x,x) = 0$,
(b) (diagonal condition) If $x, y \in X$ and $d(x,y) = 0$ then $x = y$,
(c) (symmetry condition) If $x, y \in X$ then $d(x,y) = d(y,x)$,
(d) (the triangle inequality) If $x, y, z \in X$ then $d(x,y) \leq d(x,z) + d(z,y)$.

The goal of this section is to explain that if (X, d) is a strict metric space then

$$\begin{array}{ccccc} \text{cover compact} & \Longrightarrow & \text{ball compact} & \Longrightarrow & \text{bounded} \\ \Downarrow\ \Uparrow & & & & \\ \text{sequentially compact} & \Longrightarrow & \text{Cauchy compact} & \Longrightarrow & \text{closed in } X \end{array} \tag{MC}$$

The properties appearing in (MC) are defined below. *Sequential compactness* is a condition stated purely in terms of sequences and *cover compactness* is a condition stated purely in terms of open sets (topology) and so, a priori, it is amazing that they turn out to be equivalent.

In addition to (MC) the following results are of importance.

Theorem 4.1. *Let (X, d) be a metric space and let $A \subseteq X$. If A is ball compact and Cauchy compact then A is cover compact.*

$$\begin{array}{ccccc} \textit{cover compact} & \Longrightarrow & \textit{ball compact} & \Longrightarrow & \textit{bounded} \\ \Downarrow\ \Uparrow & \Longleftarrow & + & & \\ \textit{sequentially compact} & \Longrightarrow & \textit{Cauchy compact} & \Longrightarrow & \textit{closed in } X \end{array}$$

Theorem 4.2. *Let $X = \mathbb{R}^n$ with metric given by $d(x,y) = |x - y|$. Let $A \subseteq X$. If A is closed in X and bounded then A is cover compact.*

$$\begin{array}{ccccc} \textit{cover compact} & \Longrightarrow & \textit{ball compact} & \overset{\text{in } \mathbb{R}^n}{\Longleftrightarrow} & \textit{bounded} \\ \Downarrow\ \Uparrow & \Longleftarrow & + & & \\ \textit{sequentially compact} & \Longrightarrow & \textit{Cauchy compact} & \overset{\text{in } \mathbb{R}^n}{\Longleftrightarrow} & \textit{closed in } X \end{array}$$

4.1.1 Definitions of the types of compactness

The set

$$\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\} = \{10^{-k} \mid k \in \mathbb{Z}_{>0}\} \qquad \text{is the } \textit{accuracy set}.$$

Specifying an element of $\mathbb{E}$ specifies the desired number of decimal places of accuracy.

Let (X, d) be a strict metric space.

A *sequence in X* is

$$\text{a function} \qquad \begin{array}{rccc} \vec{x}\colon & \mathbb{Z}_{>0} & \to & X \\ & n & \mapsto & x_n \end{array}$$

Let $(x_1, x_2, \ldots)$ be a sequence in X. A *limit point of* $(x_1, x_2, \ldots)$ is $z \in X$ such that

$$\text{if } \epsilon \in \mathbb{E} \text{ then there exists } \ell \in \mathbb{Z}_{>0} \text{ such that } \quad \text{if } n \in \mathbb{Z}_{\ge \ell} \text{ then } d(x_n, z) < \epsilon.$$

Write $z = \lim_{k\to\infty} x_k$ if z is a limit point of $(x_1, x_2, \ldots)$.
A *cluster point of* $(x_1, x_2, \ldots)$ is $z \in X$ such that

$$\text{there exists a subsequence } (x_{n_1}, x_{n_2}, \ldots) \text{ of } (x_1, x_2, \ldots) \text{ such that } \quad z = \lim_{k\to\infty} x_{n_k}.$$

A *Cauchy sequence* in X is a sequence $(x_1, x_2, \ldots)$ in X such that

$$\text{if } \epsilon \in \mathbb{E} \quad \text{then there exists } \ell \in \mathbb{Z}_{>0} \text{ such that } \qquad \text{if } m, n \in \mathbb{Z}_{\ge \ell} \text{ then } d(x_m, x_n) < \epsilon.$$

Let $x \in X$ and let $\epsilon \in \mathbb{E}$. The *open ball of radius* ϵ *at* x is

$$B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}.$$

The *metric space topology on* X is

$$\mathcal{T}_d = \{U \subseteq X \mid \text{if } x \in U \text{ then there exists } \epsilon \in \mathbb{E} \text{ such that } B_\epsilon(x) \subseteq U\}.$$

An *open set in* X is a set in $\mathcal{T}_d$.

Let (X, d) be a strict metric space and let $A \subseteq X$.

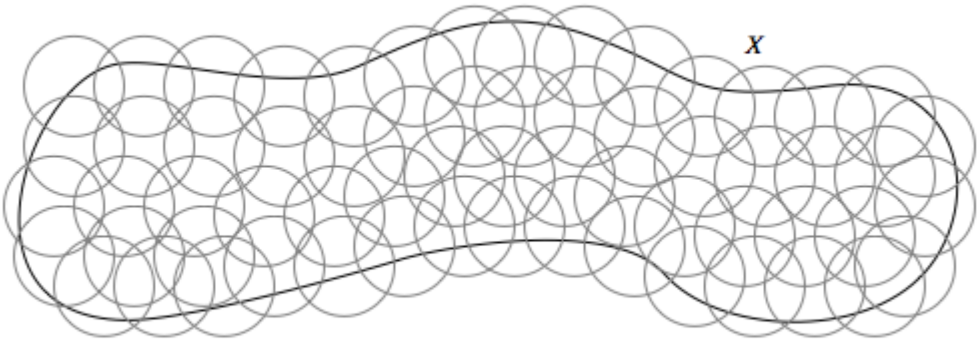
- The set A is *sequentially compact* if every sequence in A has a cluster point in A.
- The set A is *Cauchy compact*, or *complete*, if every Cauchy sequence in A has a limit point in A. (In English: A is complete if every Cauchy sequence in A converges in A.)
- The set A is *closed in* X if A satisfies:
  if $(a_1, a_2, \ldots)$ is a sequence in A and $z \in X$ is a limit point of $(a_1, a_2, \ldots)$ then $z \in A$.
  (In English: A is closed if every limit point for A is in A.)
- The set A is *bounded* if there exists $x_1 \in X$ and $\epsilon \in \mathbb{R}_{>0}$ such that $A \subseteq B_\epsilon(x_1)$.
  (In English: A is bounded if can be covered by a single open ball.)
- The set A is *ball compact in* X if A satisfies
  if $\epsilon \in \mathbb{E}$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in X$ such that

  $$A \subseteq B_\epsilon(x_1) \cup B_\epsilon(x_2) \cup \cdots B_\epsilon(x_\ell).$$

  (In English: A can be covered by a finite number of balls of radius ϵ.)
- The set A is *cover compact* if A satisfies

  $$\begin{aligned}&\text{if } \mathcal{S} \subseteq \mathcal{T}_d \text{ and } A \subseteq \Big(\bigcup_{U\in\mathcal{S}} U\Big) \text{ then there exists } \ell \in \mathbb{Z}_{>0} \text{ and} \\ &U_1, U_2, \ldots, U_\ell \in \mathcal{S} \text{ such that } A \subseteq U_1 \cup U_2 \cup \cdots \cup U_\ell.\end{aligned} \tag{C4}$$

  (In English: every open cover has a finite subcover.)
- Synonyms for *ball compact* are *precompact* and *totally bounded*.



cartoon of the finite subcover property (C4)

4.1.2 Cluster and limit points; Cauchy and convergent sequences

The proof of part (e) of Proposition 4.3 uses Proposition 4.4(a).

Proposition 4.3. *Let (X, d) be a metric space. Let $A \subseteq X$ and let $(a_1, a_2, \ldots)$ be a sequence in A.*

(a) (Limit points are unique) If $z_1, z_2 \in X$ are limit points of $(a_1, a_2, \ldots)$ then $z_1 = z_2$.

(b) (Limit points are cluster points) If $z \in X$ is a limit point of $(a_1, a_2, \ldots)$ then z is a cluster point of $(a_1, a_2, \ldots)$.

(c) (Cluster points of Cauchy sequences are limit points) If $(a_1, a_2, \ldots)$ is a Cauchy sequence and z is a cluster point of $(a_1, a_2, \ldots)$ then z is a limit point of $(a_1, a_2, \ldots)$.

(d) (Convergent sequences are Cauchy) If there exists $z \in X$ such that z is a limit point of $(a_1, a_2, \ldots)$ then $(a_1, a_2, \ldots)$ is Cauchy sequence.

(e) If A is ball compact in X then $(a_1, a_2, \ldots)$ has a Cauchy subsequence.

4.1.3 Compactness and subspaces

Proposition 4.4. *Let (X, d) be a metric space and let $A \subseteq X$. Let $B \subseteq A$.*

(a) If B is ball compact in A then B is ball compact in B.

(b) If A is bounded then B is bounded.

(c) If A is cover compact and B is closed in A then B is cover compact.

(d) If A is sequentially compact and B is closed in A then B is sequentially compact.

(e) If A is Cauchy compact and B is closed in A then B is Cauchy compact.

Sketch of proof of Proposition 4.4.
(a) Since $A \supseteq B$ then a finite cover of A by $B_\epsilon(x)$ is a finite cover of B by $B_\epsilon(x)$.
(b) Since $A \supseteq B$ then $B_M(x) \supseteq A$ implies $B_M(x) \supseteq B$.
(c) If $(b_1, b_2, \ldots)$ is a sequence in B then it is also a sequence in A and since B is closed a cluster point of $(b_1, b_2, \ldots)$ in A will also be in B.
(d) If $(b_1, b_2, \ldots)$ is a Cauchy sequence in B then it is also a sequence in A and since B is closed a limit point of $(b_1, b_2, \ldots)$ in A will also be in B.

4.1.4 The additional statements in Theorem 4.2

Proposition 4.5. *Let $\mathbb{R}^n$ have the standard metric and let $A \subseteq \mathbb{R}^n$.*

(a) (Bounded subsets of $\mathbb{R}^n$ are ball compact) If A is bounded then A is ball compact.

(b) (Closed subsets of $\mathbb{R}^n$ are Cauchy compact) If A is closed in $\mathbb{R}^n$ then A is Cauchy compact.

Proof. *(Sketch)* If A is bounded of diameter M then (by the Archimedean property of $\mathbb{R}$) A can be covered with a finite number (about $10^\ell M$) of cubes of width $10^{-\ell}$. Since $\mathbb{R}^n$ is complete part (b) follows from Proposition 4.4(e).

4.1.5 Sketch of the proof of the implications in (MC)

Cover compact $\Rightarrow$ ball compact: If $\varepsilon \in \mathbb{E}$ then $\mathcal{S}_\varepsilon = \{B_\varepsilon(a) \mid a \in A\}$ is a cover of A.

Ball compact $\Rightarrow$ bounded: If A is ball compact and $A \subseteq B_1(x_1) \cup \cdots B_1(x_\ell)$ then $x = x_1$ and $M = \max\{d(x_1, x_2), d(x_1, x_3), \ldots, d(x_1, x_\ell)\} + 2$ should do to get $A \subseteq B_M(x)$.

Cover compact $\Rightarrow$ sequentially compact: Assume $(a_1, a_2, \ldots)$ is a sequence with no cluster point. For $x \in X$ let U_x be open such that $x \in U_x$ and U_x does not contain all but a finite number of points of $(a_1, a_2, \ldots)$. Then $\mathcal{S} = \{U_x \mid x \in X\}$ is an open cover with no finite subcover.

Sequentially compact $\Rightarrow$ Cauchy compact: If $(a_1, a_2, \ldots)$ is a Cauchy sequence, it has a cluster point since A is seq. compact, and this cluster point is a limit point since $(a_1, a_2, \ldots)$ is Cauchy.

Cauchy compact $\Rightarrow$ closed: If A is Cauchy compact and z is a close point to A then there is a sequence $(a_1, a_2, \ldots)$ that converges to z. Since convergent sequences are Cauchy, this is a Cauchy sequence in A. Since A is Cauchy compact, its limit point z is in A.

Sequentially compact $\Rightarrow$ ball compact: Assume A is not ball compact. Let $\epsilon \in \mathbb{E}$ such that $\mathcal{S}_\epsilon = \{B_\epsilon(a) \mid a \in A\}$ does not have a finite subcover. Let $a_1, a_2, \ldots \in A$ such that

$$a_1 \in A \quad \text{and} \quad a_{k+1} \in (B_\epsilon(a_1) \cup \cdots \cup B_\epsilon(a_{k-1}))^c.$$

Then $(a_1, a_2, \ldots)$ is a sequence in A which has no cluster point since $d(a_i, a_j) \geq \epsilon$ for $i \neq j$.

Ball compact and Cauchy compact $\Rightarrow$ sequentially compact: Assume $(a_1, a_2, \ldots)$ is a sequence in A. Since A is ball compact $(a_1, a_2, \ldots)$ has a Cauchy subsequence which converges in A, since A is Cauchy compact. This limit point is a cluster point of $(a_1, a_2, \ldots)$.

Ball compact and Cauchy compact $\Rightarrow$ cover compact: Assume A is ball compact and not cover compact. Let $\mathcal{S}$ be an open cover of A with no finite subcover. Using that A is ball compact, choose a "bad ball" (there will be at least one) $B_{10^{-1}}(a_1)$ of radius 10^{-1} from a finite 10^{-1}-cover of A, i.e. $B_{10^{-1}}(a_1) \cap A$ cannot be covered by a finite subset of $\mathcal{S}$. Next pick a "bad ball" (there will be at least one) $B_{10^{-2}}(a_2)$ of radius 10^{-2} from a finite 10^{-2}-cover of $A \cap B_{10^{-1}}(a_1)$, i.e. $B_{10^{-2}}(a_2) \cap B_{10^{-1}}(a_1) \cap A$ cannot be covered by a finite subset of $\mathcal{S}$. Continue this process to produce a Cauchy sequence $(a_1, a_2, \ldots)$ in A. This Cauchy sequence does not have a limit point. So A is not Cauchy compact.

4.2 Hausdorff and compact topological spaces

4.2.1 Topological spaces

A *topological space* is a set X with a specification of the *open* subsets of X where it is required that

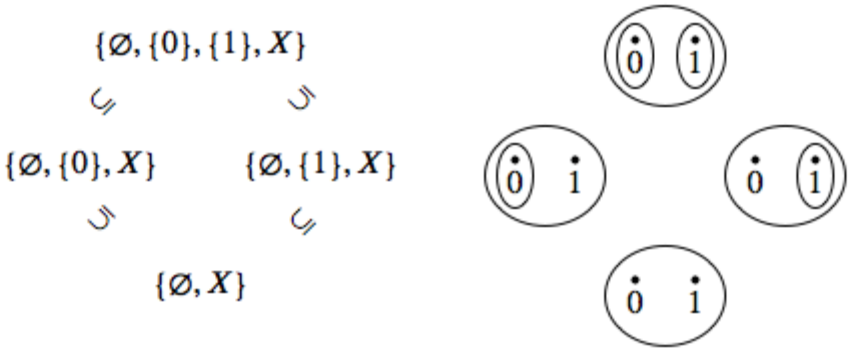
(a) $\emptyset$ is open in X and X is open in X,
(b) Unions of open sets in X are open in X,

(c) Finite intersections of open sets in X are open in X.

In other words, a *topology* on X is a set $\mathcal{T}$ of subsets of X such that

(a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
(b) If $\mathcal{S} \subseteq \mathcal{T}$ then $\left(\bigcup_{U\in\mathcal{S}} U\right) \in \mathcal{T}$,
(c) If $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{T}$ then $U_1 \cap U_2 \cap \cdots \cap U_\ell \in \mathcal{T}$.

A *topological space* is a set X with a topology $\mathcal{T}$ on X. An *open set in* X is a set in $\mathcal{T}$.

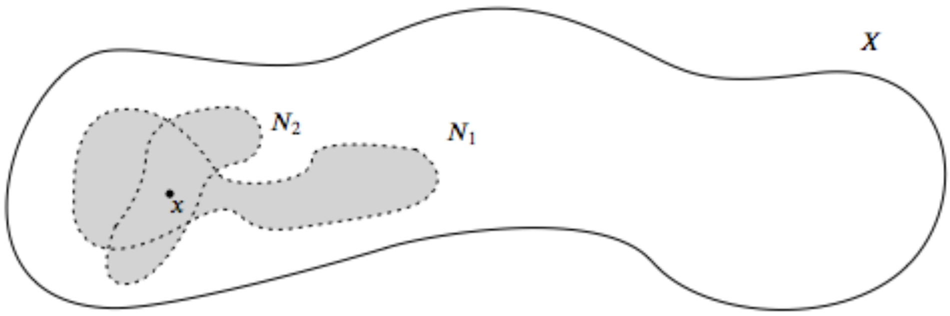


The four possible topologies on $X = \{0, 1\}$.

In a topological space, perhaps even more important than the open sets are the neighborhoods. Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$. The *neighborhood filter of* x is

$$\mathcal{N}(x) = \{N \subseteq X \mid \text{there exists } U \in \mathcal{T} \text{ such that } x \in U \text{ and } N \supseteq U\}.$$

A *neighborhood of* x is a set in $\mathcal{N}(x)$.



Neighborhoods of x.

Let $(X, \mathcal{T})$ be a topological space.

A *closed set in* X is $K \subseteq X$ such that the complement $X - K$ is open.

Let $A \subseteq X$. A *close point* to A is an element $x \in X$ such that

$$\text{if } N \in \mathcal{N}(x) \text{ then } N \cap A \neq \emptyset.$$

The *closure of* A is the subset $\overline{A}$ of X such that

(a) $\overline{A}$ is closed in X and $\overline{A} \supseteq A$,
(b) If C is closed in X and $C \supseteq A$ then $C \supseteq \overline{A}$.

Proposition 4.6. *Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. The closure of A is the set of close points of A.*

4.3 Filters

Let X be a set. A *filter on* X is a collection $\mathcal{F}$ of subsets of X such that

(a) $\emptyset \notin \mathcal{F}$.
(b) (upper ideal) If $N \in \mathcal{F}$ and E is a subset of X with $N \subseteq E$ then $E \in \mathcal{F}$,
(c) (closed under finite intersection) If $\ell \in \mathbb{Z}_{>0}$ and

$$N_1, N_2, \ldots, N_\ell \in \mathcal{F} \qquad \text{then} \qquad N_1 \cap N_2 \cap \cdots \cap N_\ell \in \mathcal{F},$$

An *ultrafilter on* X is a maximal filter on X, i.e. an ultrafilter on X is a filter $\mathcal{G}$ on X such that

$$\text{if } \mathcal{F} \text{ is a filter on } X \text{ and } \mathcal{F} \supseteq \mathcal{G} \text{ then } \mathcal{F} = \mathcal{G}.$$

Let $(X, \mathcal{T})$ be a topological space and let $z \in Z$. The *neighborhood filter of* z is

$$\mathcal{N}(z) = \{N \subseteq X \mid \text{there exists } U \in \mathcal{T} \text{ such that } x \in U \text{ and } N \supseteq U\}$$

Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{F}$ be a filter on X.

A *limit point of* $\mathcal{F}$ is $z \in X$ such that $\mathcal{F} \supseteq \mathcal{N}(z)$.

A *cluster point of* $\mathcal{F}$ is $z \in X$ such that there exists a filter $\mathcal{G}$ on X with $\mathcal{G} \supseteq \mathcal{F}$ and z is a limit point of $\mathcal{G}$.

4.4 Hausdorff topological spaces

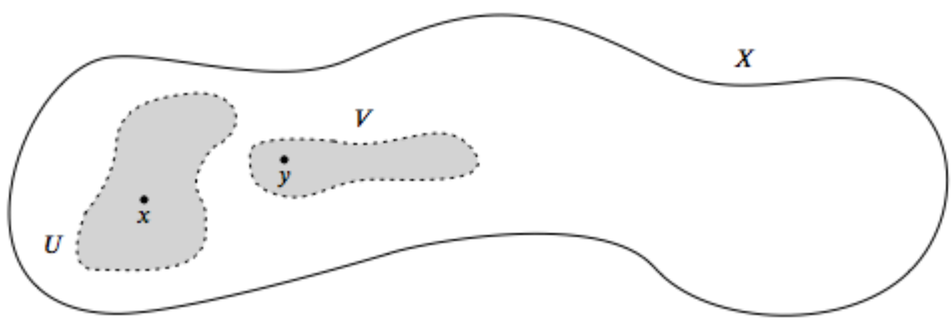
The goal of this section is to explain that if $(X, \mathcal{T})$ is a topological space then

$$\text{limit unique} \quad \Leftrightarrow \quad \text{Hausdorff} \quad \Leftrightarrow \quad \text{separated} \quad \Leftrightarrow \quad \text{neighborhood pinpointed} \qquad \text{(H)}$$

The definitions of these terms are as follows. Let $(X, \mathcal{T})$ be a topological space.

- The space $(X, \mathcal{T})$ is *limit unique* if every filter on X has at most one limit point.
- The space $(X, \mathcal{T})$ is *Hausdorff* if $(X, \mathcal{T})$ satisfies

$$\text{if } x, y \in X \text{ and } x \neq y \text{ then there exists } U \in \mathcal{N}(x) \text{ and } V \in \mathcal{N}(y) \text{ such that } U \cap V = \emptyset,$$



The Hausdorff property

- The space $(X, \mathcal{T})$ is *separated* if

$$\Delta(X) = \{(x, x) \mid x \in X\} \text{ is a closed subset of } X \times X$$

(with the product topology on $X \times X$).

- The space $(X, \mathcal{T})$ is *neighborhood pinpointed* if $(X, \mathcal{T})$ satisfies

$$\text{if } x \in X \quad \text{then} \qquad \bigcap_{N \in \mathcal{N}(x)} \overline{N} = \{x\}.$$

4.4.1 Sketch of the proof of the equivalences in (H)

Theorem 4.7. *The following conditions on a topological space $(X, \mathcal{T})$ are equivalent.*

(limit unique) If $\mathcal{G}$ is a filter on X then $\mathcal{G}$ has at most one limit point.

(cluster unique) If $\mathcal{F}$ is a filter on X and x is a limit point of $\mathcal{F}$ then x is the only cluster piont of $\mathcal{F}$.

(neighborhood pinpointed) If $x \in X$ then $\displaystyle\bigcap_{N \in \mathcal{N}(x)} \overline{N} = \{x\}$.

(Hausdorff) If $x, y \in X$ and $x \neq y$ then there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V = \emptyset$.

(separated) $\Delta(X) = \{(x, x) \mid x \in X\}$ is a closed subset of $X \times X$ (with the product topology on $X \times X$).

Sketch of proof.

Hausdorff $\Leftrightarrow$ separated: The point here is that if $x, y \in X$ with $x \neq y$ then $(x, y) \in X \times X$ is not a close point to $\Delta(X)$ if and only if there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $(U \times V) \cap \Delta(X) = \emptyset$ and this happens if and only if there exists $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V = \emptyset$.

Hausdorff $\Leftrightarrow$ neighborhood pinpointed $\Leftrightarrow$ (H6) $\Leftrightarrow$ limit unique: The point here is that if $(X, \mathcal{T})$ is Hausdroff holds and $x, y \in X$ with $y \neq x$ then $y \notin \bigcap_{U \in \mathcal{N}(x)} \overline{U}$ which is equivalent to $\{x\} = \bigcap_{U \in \mathcal{N}(x)} \overline{U}$ so that x is the only cluster point of $\mathcal{N}(x)$. If $\mathcal{F}$ is a filter with x as a limit point then x is also a cluster point of $\mathcal{F}$ and

$$x \in \bigcap_{M \in \mathcal{F}} \overline{M} \subseteq \bigcap_{U \in \mathcal{N}(x)} \overline{U} = \{x\}.$$

□

4.5 Compact topological spaces

The goal of this section is to explain that if $(X, \mathcal{T})$ is a topological space then

$$\text{filter compact} \;\Leftrightarrow\; \text{ultrafilter compact} \;\Leftrightarrow\; \text{exclusion compact} \;\Leftrightarrow\; \text{cover compact} \qquad \text{(C)}$$

The definitions of these terms are as follows. Let $(X, \mathcal{T})$ be a topological space.

- The space $(X, \mathcal{T})$ is *filter compact* if every filter has a cluster point.
- The space $(X, \mathcal{T})$ is *ultrafilter compact* if every ultrafilter has a limit point.
- The space $(X, \mathcal{T})$ is *exclusion compact* if every closed exclusion contains a finite exclusion, i.e.

  If $\mathcal{C}$ is a collection of closed sets of X such that $\displaystyle\bigcap_{K \in \mathcal{C}} K = \emptyset$

  then there exists $\ell \in \mathbb{Z}_{>0}$ and $K_1, K_2, \ldots, K_\ell \in \mathcal{C}$ such that $K_1 \cap K_2 \cap \cdots \cap K_\ell = \emptyset$.

- The space $(X, \mathcal{T})$ is *cover compact* if every open cover has a finite subcover, i.e.

  If $\mathcal{S}$ is a collection of open sets of X such that $\bigcup_{U\in\mathcal{S}} U = X$
  then there exists $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{S}$ such that $U_1 \cup U_2 \cup \cdots \cup U_\ell = X$.

4.5.1 Sketch of the proof of the equivalences in (C)

Theorem 4.8. *The following conditions on a topological space $(X, \mathcal{T})$ are equivalent.*

(C1: filter compact) If $\mathcal{F}$ is an filter on X then there exists $x \in X$ such that x is a cluster point of $\mathcal{F}$.

(C2: ultrafilter compact) If $\mathcal{G}$ is an ultrafilter on X then there exists $x \in X$ such that x is a limit point of $\mathcal{G}$.

(C3: exclusion compact) If $\mathcal{C}$ is a collection of closed sets of X such that $\bigcap_{K\in\mathcal{C}} K = \emptyset$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $K_1, K_2, \ldots, K_\ell \in \mathcal{C}$ such that $K_1 \cap K_2 \cap \cdots \cap K_\ell = \emptyset$.

(C4: cover compact) If $\mathcal{S}$ is a collection of open sets of X such that $\bigcup_{U\in\mathcal{S}} U = X$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{S}$ such that $U_1 \cup U_2 \cup \cdots \cup U_\ell = X$.

Sketch of proof.
exclusion compact $\Leftrightarrow$ cover compact: By taking complements.

(not exclusion compact) $\Leftrightarrow$ (not filter compact): Produce a filter $\mathcal{F}$ with no cluster point from a collection of closed sets $\mathcal{C}$ with does not satisfy (C3) by letting

$$\mathcal{F} = \{N \subseteq X \mid \text{there exists } K \in \mathcal{C} \text{ with } N \supseteq K\}$$

and produce a set of closed sets $\mathcal{C}$ that does not satisfy (C3) from a filter $\mathcal{F}$ with no cluster point by setting

$$\mathcal{C} = \{\overline{N} \mid N \in \mathcal{F}\}.$$

filter compact $\Leftrightarrow$ ultrafilter compact: The point is that every filter $\mathcal{F}$ is contained an ultrafilter $\mathcal{G}$ and every cluster point of an ultrafilter is a limit point. □

4.5.2 Cover compactness and subspaces

Proposition 4.9. *Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$.*

(a) If X is Hausdorff and A is cover compact then A is closed in X.

(b) Let $B \subseteq A$. If A is cover compact and B is closed in X then B is cover compact.

4.6 Notes and references

Theorem 4.10. *(see [Ra1]) Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$. Then*

$$\overline{A} = \big\{z \in X \mid \textit{there exists a sequence } (a_1, a_2, \ldots) \textit{ in } A \textit{ such that } z = \lim_{n\to\infty} a_n\big\},$$

where $\overline{A}$ is the closure of A in X.

4.6.1 Spaces

Although it is traditional to define topological spaces via axioms for **open sets**, there are equivalent (and useful!) definitions of topological spaces by axioms for the **closed sets**, and via axioms for **neighborhoods**. Another important and useful point of view is to view the topological spaces as a category with morphisms the **continuous functions**. From this point of view the notion of *topological space* and the notion of *continuous function* are "equivalent data".

4.6.2 Filters, Hausdorff and Compact spaces

The treatment of filters here is a distillation of material found in Bourbaki: the definition of filter, is in [Bou, Top. Ch. I §6 no. 1], the definition of limit point and cluster point of a filter are [Bou, Top. Ch. I, §7 Def. 1 and 2] and the definition of limit point and cluster point of a function are [Bou, Top. Ch. I §7 Def. 3]. Theorem **??** is [Bou, Top. Ch. I §7 Prop. 9] and Proposition **??** is Example 1 in [Bou, Top. Ch. I §7 no. 3].

The presentation of the equivalent conditions for **Hausdorff spaces**, Theorem 4.7, follows Bourbaki [Bou, Top. Ch. I §8 no. 1].

(H3) The condition that $\Delta(X)$ is closed in $X \times X$ is the condition used in algebraic geometry for a separated scheme (see [Ha, Ch. II §4] and Macdonald (1.11) in [CSM]).

(H5) Hausdorff spaces are the spaces such that limits are unique, when they exist.

(H1) The condition (H1) is the separation axiom that is used often as the definition of a Hausdorff topological space.

The presentation of the equivalent conditions for **compact spaces**, Theorem 4.8, follows [Bou, Top. Ch. I §9 no. 1]. The second and third conditions in the definition of a filter say that finite intersections of elements of a filter cannot be empty. This is the rigidity condition that plays an important role in arguments relating limit points and compactness.

4.7 Some proofs

4.7.1 Cluster and limit points; Cauchy and convergent sequences

The proof of part (e) of Proposition 4.11 uses Proposition 4.12(a).

Proposition 4.11. *Let (X, d) be a metric space. Let $A \subseteq X$ and let $(a_1, a_2, \ldots)$ be a sequence in A.*

(a) (Limit points are unique) If $z_1, z_2 \in X$ are limit points of $(a_1, a_2, \ldots)$ then $z_1 = z_2$.

(b) (Limit points are cluster points) If $z \in X$ is a limit point of $(a_1, a_2, \ldots)$ then z is a cluster point of $(a_1, a_2, \ldots)$.

(c) (Cluster points of Cauchy sequences are limit points) If $(a_1, a_2, \ldots)$ is a Cauchy sequence and z is a cluster point of $(a_1, a_2, \ldots)$ then z is a limit point of $(a_1, a_2, \ldots)$.

(d) (Convergent sequences are Cauchy) If there exists $z \in X$ such that z is a limit point of $(a_1, a_2, \ldots)$ then $(a_1, a_2, \ldots)$ is Cauchy sequence.

(e) If A is ball compact in X then $(a_1, a_2, \ldots)$ has a Cauchy subsequence.

Proof.

(a) Assume $z_1, z_2 \in X$ are limit points of $(a_1, a_2, \ldots)$.
To show: $z_1 = z_2$.
To show: $d(z_1, z_2) = 0$.
To show: If $\epsilon \in \mathbb{E}$ then $d(z_1, z_2) < \epsilon$.
Assume $\epsilon \in \mathbb{E}$.
To show: $d(z_1, z_2) < \epsilon$.
Let $\ell_1 \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell_1}$ then $d(a_n, z_1) < \frac{1}{2}\epsilon$.
Let $\ell_2 \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell_2}$ then $d(a_n, z_2) < \frac{1}{2}\epsilon$.
Let $\ell = \max(\ell_1, \ell_2)$.
By the triangle inequality,

$$d(z_1, z_2) \leq d(z_1, a_\ell) + d(a_\ell, z_2) < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.$$

So $d(z_1, z_2) = 0$.
So $z_1 = z_2$.

(b) Assume z is a limit point of $(a_1, a_2, \ldots)$.
Thus, if $\epsilon \in \mathbb{E}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $d(a_n, z) < \epsilon$.
To show: There exists a subsequence $(a_{n_1}, a_{n_2}, \ldots)$ of $(a_1, a_2, \ldots)$ such that $z = \lim_{k\to\infty} a_{n_k}$.
Let $n_1 = 1$, $n_2 = 2$, ..., so that $n_k = k$.
To show: $\lim_{k\to\infty} a_{n_k} = z$.
Since $\lim_{k\to\infty} a_k = z$ then

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} a_k = z.$$

So z is a cluster point of $(a_1, a_2, \ldots)$.

(c) Let $(a_1, a_2, \ldots)$ be a Cauchy sequence in X and let z be a cluster point of $(a_1, a_2, \ldots)$.
To show: $z = \lim\limits_{n\to\infty} a_n$.
To show: If $\epsilon \in \mathbb{E}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $d(a_n, z) < \epsilon$.
Assume $\epsilon \in \mathbb{E}$.
To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $d(a_n, z) < \epsilon$.
Since $(a_1, a_2, \ldots)$ is a Cauchy sequence then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, k \in \mathbb{Z}_{\geq N}$ then $d(a_m, a_k) < \frac{1}{2}\epsilon$.
Since z is a cluster point of $a_1, a_2, \ldots$ then there exists a subsequence $(a_{m_1}, a_{m_2}, \ldots)$ such that $\lim\limits_{p\to\infty} a_{m_p} = z$.
So there exists $m_p \in \mathbb{Z}_{\geq N}$ such that $d(a_{m_p}, z) < \frac{1}{2}\epsilon$.
To show: If $n \in \mathbb{Z}_{\geq N}$ then $d(a_n, z) < \epsilon$.
Assume $n \in \mathbb{Z}_{\geq N}$.
To show: $d(a_n, z) < \epsilon$.

$$d(a_n, z) \leq d(a_n, a_{m_p}) + d(a_{m_p}, z) < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.$$

So $z = \lim_{n\to\infty} a_n$.

(d) Let $(a_1, a_2, \ldots)$ be a convergent sequence in X.
Then there exists $z \in X$ such that $\lim\limits_{k\to\infty} a_k = z$.

To show: $(a_1, a_2, \ldots)$ is a Cauchy sequence.
To show: If $\epsilon \in \mathbb{E}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\geq \ell}$ then $d(a_m, a_n) < \epsilon$.
Assume $\epsilon \in \mathbb{E}$.
Since $\lim_{k\to\infty} a_k = z$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $m \in \mathbb{Z}_{\geq \ell}$ then $d(a_m, z) < \frac{\epsilon}{2}$.
To show: If $m, n \in \mathbb{Z}_{\geq \ell}$ then $d(a_m, a_n) < \epsilon$.
Assume $m, n \in \mathbb{Z}_{\geq \ell}$.
To show: $d(a_m, a_n) < \epsilon$.

$$d(a_m, a_n) \leq d(a_m, z) + d(z, a_n) < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.$$

So $d(a_m, a_n) < \epsilon$.
So $(a_1, a_2, \ldots)$ is a Cauchy sequence.

(e) Assume A is ball compact and $(a_1, a_2, \ldots)$ is a sequence in A.
To show: There exists a subsequence $(a_{n_1}, a_{n_2}, \ldots)$ of $(a_1, a_2, \ldots)$ which is Cauchy.
Since A is ball compact in A then $\{a_1, a_2, \ldots\}$ is ball compact in A.
By Proposition 4.4(a), then $\{a_1, a_2, \ldots\}$ is ball compact in $\{a_1, a_2, \ldots\}$.
Since $\{a_1, a_2, \ldots\}$ is ball compact in $\{a_1, a_2, \ldots\}$ then if $\epsilon \in \mathbb{R}_{>0}$ then there exists $n \in \mathbb{Z}_{>0}$ such that $B_\epsilon(x_n)$ contains an infinite number of $(a_1, a_2, \ldots)$.
Using the pigeonhole principle (if you have a finite number of boxes containing an infinite number of pigeons then at least one box will contain an infinite number of pigeons),

> Let $n_1 \in \mathbb{Z}_{>0}$ be minimal such that $B_1(a_{n_1})$ contains an infinite number of $(a_1, a_2, \ldots)$;
> Let $n_2 \in \mathbb{Z}_{>n_1}$ be minimal such that $B_{\frac{1}{2}}(a_{n_2})$ contains an infinite number of $(a_1, a_2, \ldots) \cap B_1(a_{n_1})$;
> Let $n_3 \in \mathbb{Z}_{>n_2}$ be minimal such that $B_{\frac{1}{3}}(a_{n_3})$ contains an infinite number of $(a_1, a_2, \ldots) \cap B_1(a_{n_1}) \cap B_{\frac{1}{2}}(a_{n_2})$;
> etc.

To show: $(a_{n_1}, a_{n_2}, \ldots)$ is Cauchy.
To show: If $\epsilon \in \mathbb{E}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(a_{n_r}, a_{n_s}) < \epsilon$.
Assume $\epsilon \in \mathbb{E}$.
Let $\epsilon = 10^{-k}$. Let $\ell = 10^{k+1}$ so that $\dfrac{1}{\ell} = 10^{-k+1} < \dfrac{\epsilon}{2}$.
To show: If $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(a_{n_r}, a_{n_s}) < \epsilon$.
Assume $r, s \in \mathbb{Z}_{\geq \ell}$.
To show: $d(a_{n_r}, a_{n_s}) < \epsilon$.
Let $a_p \in (a_1, a_2, \ldots) \cap (B_1(a_{n_1}) \cap \cdots \cap B_{\frac{1}{r}}(a_{n_r})) \cap (B_1(a_{n_1}) \cap \cdots \cap B_{\frac{1}{s}}(a_{n_s}))$.
Then

$$d(a_{n_r}, a_{n_s}) \leq d(a_{n_r}, a_{n_\ell}) + d(a_{n_\ell}, a_{n_s}) < \frac{1}{r} + \frac{1}{s} < \frac{1}{\ell} + \frac{1}{\ell} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So $(a_{n_1}, a_{n_2}, \ldots)$ is Cauchy.

□

4.7.2 Compactness and subspaces

Proposition 4.12. *Let (X,d) be a metric space and let $A \subseteq X$. Let $B \subseteq A$.*

(a) If B is ball compact in A then B is ball compact in B.

(b) If A is bounded then B is bounded.

(c) If A is cover compact and B is closed in A then B is cover compact.

(d) If A is sequentially compact and B is closed in A then B is sequentially compact.

(e) If A is Cauchy compact and B is closed in A then B is Cauchy compact.

Proof.

(a) Assume B is ball compact in A.
To show: B is ball compact in B.
To show: If $\epsilon \in \mathbb{E}$ then there exist $\ell \in \mathbb{Z}_{>0}$ and $b_1, b_2, \ldots, b_\ell \in B$ such that $B_\epsilon(b_1) \cup B_\epsilon(b_2) \cup \cdots \cup B_\epsilon(b_\ell) \supseteq B$.
Assume $\epsilon \in \mathbb{E}$.
To show: There exist $\ell \in \mathbb{Z}_{>0}$ and $b_1, b_2, \ldots, b_\ell \in B$ such that $B_\epsilon(b_1) \cup B_\epsilon(b_2) \cup \cdots \cup B_\epsilon(b_\ell) \supseteq B$.
Using that B is ball compact in A, let $m \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_m \in A$ such that

$$B_{\epsilon/2}(a_1) \cup \cdots \cup B_{\epsilon/2}(a_m) \supseteq B.$$

Let $J = \{j \in \{1, \ldots, m\} \mid B_{\epsilon/2}(a_j) \cap B \neq \emptyset\}$ and let $\ell = \mathrm{Card}(J)$.
For $j \in J$ let $b_j \in B_{\epsilon/2}(a_j) \cap B \neq \emptyset$.
Then, since $B_\epsilon(b_j) \supseteq B_{\frac{\epsilon}{2}}(a_j)$,

$$\Big(\bigcup_{j \in J} B_\epsilon(b_j)\Big) \supseteq (B_{\frac{\epsilon}{2}}(a_1) \cap B) \cup \cdots \cup (B_{\frac{\epsilon}{2}}(a_m) \cap B) \supseteq B \cap B = B.$$

So B is ball compact in B.

(b) Assume A is bounded.
To show: B is bounded.
Since A is bounded there exists $x_1 \in X$ and $\epsilon \in \mathbb{R}_{>0}$ such that $B_\epsilon(x_1) \supseteq A$.
Since $A \supseteq B$ then $B_\epsilon(x_1) \supseteq B$.
So B is bounded.

(c) Assume A is cover compact and B is closed in X.
To show: B is cover compact.
To show: If $\mathcal{S} \subseteq \mathcal{T}_d$ such that $B \subseteq \Big(\bigcup_{U \in \mathcal{S}} U\Big)$ then
there exists a finite subset $\mathcal{K}$ of $\mathcal{S}$ such that $B \subseteq \Big(\bigcup_{U \in \mathcal{K}} U\Big)$.
Let $\mathcal{S} \subseteq \mathcal{T}_d$ such that $B \subseteq \Big(\bigcup_{U \in \mathcal{S}} U\Big)$.
Since B is closed, then B^c is open and $\mathcal{S} \cup \{B^c\}$ is an open cover of A.
Since A is compact then there exists a finite subset $\mathcal{J} \subseteq \mathcal{S} \cup \{B^c\}$ such that $A \subseteq \Big(\bigcup_{U \in \mathcal{J}} U\Big)$.

Let

$$\mathcal{K} = \begin{cases} \mathcal{J}, & \text{if } B^c \notin \mathcal{J}, \\ \mathcal{J} - \{B^c\}, & \text{if } B^c \in \mathcal{J}. \end{cases}$$

Then $\mathcal{K}$ is a finite subset of $\mathcal{S}$ such that $B \subseteq \Big(\bigcup_{U \in \mathcal{K}} U \Big)$.

So B is cover compact.

(d) Assume that A is sequentially compact and B is closed in X.
To show: B is sequentially compact.
To show: If $(b_1, b_2, \ldots)$ is a sequence in B then there exists $z \in B$ such that z is a cluster point of $(b_1, b_2, \ldots)$.
Assume $(b_1, b_2, \ldots)$ is a sequence in B.
Since $B \subseteq A$ then $(b_1, b_2, \ldots)$ is a sequence in A.
Since A is sequentially compact there exists $z \in A$ such that such that z is a cluster point of $(b_1, b_2, \ldots)$.
Thus there is a subsequence $(b_{n_1}, b_{n_2}, \ldots)$ such that $z = \lim_{k \to \infty} b_{n_k}$.
Since B is closed in X and $(b_{n_1}, b_{n_2}, \ldots)$ is a sequence in B then $z \in B$.
So B is sequentially compact.

(e) Assume A is Cauchy compact and B is closed in X.
To show: B is Cauchy compact.
To show: If $(b_1, b_2, \ldots)$ is a Cauchy sequence in B then there exists $z \in B$ such that $z = \lim_{n \to \infty} b_n$.
Assume $(b_1, b_2, \ldots)$ is a Cauchy sequence in B.
Since $B \subseteq A$ then $(b_1, b_2, \ldots)$ is a Cauchy sequence in A.
Since A is Cauchy compact there exists $z \in A$ such that $z = \lim_{n \to \infty} b_n$.
Since B is closed in X and $(b_1, b_2, \ldots)$ is a sequence in B then $z \in B$.
So B is Cauchy compact.

□

4.7.3 Cover compactness and subspaces

Proposition 4.13. *Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$.*

(a) If X is Hausdorff and A is cover compact then A is closed in X.

(b) Let $B \subseteq A$. If A is cover compact and B is closed in X then B is cover compact.

Proof.

(a) Assume X is Hausdorff and A is compact.
To show: A is closed in X.
To show: A^c is open in X.
To show: If $x \in A^c$ then x is an interior point of A.
To show: There exists $U \in \mathcal{T}$ such that $x \in U$ and $U \subseteq A^c$.
Assume $x \in A^c$.
Using that X is Hausdorff,

for $y \in A$ let $U_{xy}, V_{xy} \in \mathcal{T}$ be such that $x \in U_{xy}$ and $y \in V_{xy}$ and $U_{xy} \cap V_{xy} = \emptyset$.
Then
$$\mathcal{S} = \{V_{xy} \mid y \in A\} \qquad \text{is an open cover of } A.$$
Since A is compact there exists $\ell \in \mathbb{Z}_{>0}$ and $y_1, \ldots, y_\ell \in A$ such that $K \subseteq V_{xy_1} \cup \cdots V_{xy_\ell}$.
Let $U = U_{xy_1} \cap \cdots \cap U_{xy_\ell}$.
Then $x \in U$ and
$$U \cap A \subseteq (U_{xy_1} \cap \cdots U_{xy_\ell}) \cap (V_{xy_1} \cup \cdots \cup V_{xy_\ell}) = \emptyset.$$
So $U \in \mathcal{T}$ and $x \in U$ and $U \subseteq A^c$.
So x is an interior point of A^c.
So A^c is open.
So A is closed.

(c) Assume A is cover compact and $B \subseteq A$ and B is closed in X.
To show: B is cover compact.
To show: If $\mathcal{S} \subseteq \mathcal{T}$ such that $B \subseteq \Big(\bigcup_{U \in \mathcal{S}} U\Big)$ then
there exists a finite subset $\mathcal{K}$ of $\mathcal{S}$ such that $B \subseteq \Big(\bigcup_{U \in \mathcal{K}} U\Big)$.
Let $\mathcal{S} \subseteq \mathcal{T}$ such that $B \subseteq \Big(\bigcup_{U \in \mathcal{S}} U\Big)$.
Since B is closed, then B^c is open and $\mathcal{S} \cup \{B^c\}$ is an open cover of A.
Since A is compact then there exists a finite subset $\mathcal{J} \subseteq \mathcal{S} \cup \{B^c\}$ such that $A \subseteq \Big(\bigcup_{U \in \mathcal{J}} U\Big)$.
Let
$$\mathcal{K} = \begin{cases} \mathcal{J}, & \text{if } B^c \notin \mathcal{J}, \\ \mathcal{J} - \{B^c\}, & \text{if } B^c \in \mathcal{J}. \end{cases}$$
Then $\mathcal{K}$ is a finite subset of $\mathcal{S}$ such that $B \subseteq \Big(\bigcup_{U \in \mathcal{K}} U\Big)$.
So B is cover compact.

□

4.8 Proofs of the implications in (MC)

4.8.1 cover compact ⇒ ball compact

Proposition 4.14. *Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is cover compact then A is ball compact.

Proof. To show: If A is cover compact then A is ball compact.
Assume A is cover compact.
To show: A is ball compact.
To show: If $k \in \mathbb{Z}_{>0}$ and $\epsilon = 10^{-k}$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_\ell \in A$ such that $A \subseteq B_\epsilon(a_1) \cup \cdots \cup B_\epsilon(a_\ell)$.
Assume $\epsilon \in \mathbb{E}$.
To show: There exists $\ell \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_\ell \in A$ such that $A \subseteq B_\epsilon(a_1) \cup \cdots \cup B_\epsilon(a_\ell)$.

Since A is cover compact and $\mathcal{S} = \{B_\epsilon(a) \mid a \in A\}$ is an open cover of A, there exists $\ell \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_\ell \in A$ such that $A \subseteq B_\epsilon(a_1) \cup \cdots \cup B_\epsilon(a_\ell)$.
So A is ball compact.

□

4.8.2 Ball compact ⇒ bounded

Proposition 4.15. *Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is ball compact then A is bounded.

Proof. To show: If A is ball compact then A is bounded.
Assume A is ball compact.
To show: A is bounded.
To show: There exists $a \in A$ and $M \in \mathbb{R}_{>0}$ such that $A \subseteq B_M(a)$.
Since A is ball compact there exists $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in A$ such that $A \subseteq B_1(x_1) \cup \cdots \cup B_1(x_\ell)$.
Let $a = x_1$ and let $M = 2 + \max\{d(x_1, x_1), d(x_1, x_2), d(x_1, x_3), \ldots, d(x_1, x_\ell)\}$.
To show: $A \subseteq B_M(a)$.
To show: If $x \in A$ then $d(x, a) < M$.
Assume $x \in A$.
To show: $d(x, a) < M$.
Let $j \in \{1, \ldots, \ell\}$ such that $x \in B_1(x_j)$.
Then

$$d(x, a) = d(x, x_1) \leq d(x, x_j) + d(x_j, x_1) \leq 1 + (M - 2) = M - 1 < M.$$

So $x \in B_M(a)$.
So $A \subseteq B_M(a)$.
So A is bounded.

□

4.8.3 cover compact ⇒ sequentially compact

Proposition 4.16. *Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is cover compact then A is sequentially compact.

Proof.

To show: If A is cover compact then A is sequentially compact.
To show: If A is not sequentially compact then A is not cover compact.
Assume A is not sequentially compact.
Then there exists a sequence $a_1, a_2, \ldots$ in A with no cluster point in A.
Thus, if $z \in A$ then there exists $N \in \mathcal{N}(z)$ such that $\mathrm{Card}\{j \mid a_j \in N\}$ is finite.
To show: A is not cover compact.
To show: There exists an open cover $\mathcal{S}$ of A which does not have a finite subcover.
For $x \in A$ let V_x be an open set of A such that

$$x \in V_x \qquad \text{and} \qquad \mathrm{Card}\{j \in \mathbb{Z}_{>0} \mid a_j \in V_x\} \text{ is finite.}$$

The set V_x exists since x is not a cluster point of $(a_1, a_2, a_3, \ldots)$.
Then $\mathcal{S} = \{V_x \mid x \in A\}$ is an open cover of A.

To show: $\mathcal{S}$ does not contain a finite subcover of A.
Assume $\ell \in \mathbb{Z}_{>0}$ and $V_{x_1}, V_{x_2}, \ldots, V_{x_\ell} \in \mathcal{S}$.
Let $k_j \in \mathbb{Z}_{>0}$ be such that if $n \in \mathbb{Z}_{\geq k_j}$ then $a_n \notin V_{x_j}$.
Let $k = \max\{k_1, k_2, \ldots, k_\ell\}$.
Then, if $n \in \mathbb{Z}_{>k}$ then $a_n \notin V_{x_1} \cup \cdots \cup V_{x_\ell}$.
So $A \not\subseteq V_{x_1} \cup \cdots \cup V_{x_\ell} \supseteq A$.
So $\mathcal{S}$ has no finite subcover.

So A is not cover compact.

□

4.8.4 sequentially compact $\Rightarrow$ Cauchy compact

Proposition 4.17. *Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is sequentially compact then A is Cauchy compact.

Proof. To show: If A is sequentially compact then A is Cauchy compact.
Assume A is sequentially compact.
To show: A is Cauchy compact.
To show: If $(a_1, a_2, \ldots)$ is a Cauchy sequence in A then $(a_1, a_2, \ldots)$ has a limit point in A.
Assume $(a_1, a_2, \ldots)$ is a Cauchy sequence in A.
To show: There exists $a \in A$ such that $\lim_{n\to\infty} a_n = a$.
Let a be a cluster point of $(a_1, a_2, \ldots)$ in A, which exists since A is sequentially compact.
By Proposition 4.3(b),
since $(a_1, a_2, \ldots)$ is Cauchy then the cluster point a is a limit point of $(a_1, a_2, \ldots)$.
So $a = \lim_{n\to\infty} a_n$.
So A is Cauchy compact.

□

4.8.5 Cauchy compact $\Rightarrow$ closed

Proposition 4.18. *Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is Cauchy compact then A is closed.

Proof. To show: If A is Cauchy compact then A is closed in X.
Assume $A \subseteq X$ and A is Cauchy compact.
To show: A is closed in X.
To show: If $(a_1, a_2, \ldots,)$ is a sequence in A and $(a_1, a_2, \ldots)$ converges in X then $\lim\limits_{k\to\infty} a_k \in A$.
Assume $(a_1, a_2, \ldots)$ is a sequence in A and $(a_1, a_2, \ldots)$ converges in X.
Since convergent sequences are Cauchy then $(a_1, a_2, \ldots)$ is a Cauchy sequence.
Since A is Cauchy compact and $(a_1, a_2, \ldots)$ is a Cauchy sequence in A then $(a_1, a_2, \ldots)$ converges in A.
Since limits in metric spaces are unique, $z = \lim\limits_{k\to\infty} a_k \in A$.
So A is closed in X. □

4.8.6 sequentially compact $\Rightarrow$ ball compact

Proposition 4.19. *Let (X, d) be a strict metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq X$.*

If A is sequentially compact then A is ball compact.

Proof.

To show: If A is sequentially compact then X is ball compact.
To show: Assume A is not ball compact.
To show: A is not sequentially compact.
To show: There exists a sequence $(a_1, a_2, \ldots)$ in A with no cluster point in A.
Using that A is not ball compact, let $\epsilon \in \mathbb{R}_{>0}$ such that A is not covered by finitely many $B_\epsilon(x)$.
Let

$$x_1 \in A, \quad x_2 \in B_{\frac{\epsilon}{2}}(a_1)^c \cap A, \quad x_3 \in (B_{\frac{\epsilon}{2}}(a_1) \cup B_{\frac{\epsilon}{2}}(a_2))^c \cap A, \quad \ldots.$$

Then $(a_1, a_2, \ldots)$ has no cluster point, since every $B_{\frac{\epsilon}{2}}(x)$ contains at most one point of $(a_1, a_2, \ldots)$.
So A is not sequentially compact.

□

4.8.7 Ball compact + Cauchy compact ⇒ sequentially compact

Proposition 4.20. *Let (X, d) be a complete metric space and let $\mathcal{T}_d$ be the metric space topology on X. Let $A \subseteq A$.*

if A is ball compact and Cauchy compact then *A is sequentially compact.*

Proof. Assume A is ball compact and Cauchy compact.
To show: A is sequentially compact.
To show: If $(a_1, a_2, \ldots)$ is a sequence in A then $(a_1, a_2, \ldots)$ has a cluster point in A.
Assume $(a_1, a_2, \ldots)$ is a sequence in A.
By Proposition 4.3(d), since A is ball compact then
there exists a subsequence $(a_{n_1}, a_{n_2}, \ldots)$ of $(a_1, a_2, \ldots)$ such that $(a_{n_1}, a_{n_2}, \ldots)$ is Cauchy.
Since A is Cauchy compact $(a_{n_1}, a_{n_2}, \ldots)$ has a limit point in A.
Thus $(a_1, a_2, \ldots)$ has a cluster point A.
So A is sequentially compact.

□

4.8.8 Ball compact + Cauchy compact ⇒ cover compact

Proposition 4.21. *Let (X, d) be a metric space and let $A \subseteq X$.*

If A is ball compact and Cauchy compact then *A is cover compact.*

Proof. To show: If A is Cauchy compact then A is cover compact.
To show: If A is not cover compact then A is not Cauchy compact.
Assume A is not cover compact.
Let $\mathcal{S}$ be an open cover with no finite subcover.
Let $a_1^{(1)}, \ldots, a_{\ell_1}^{(1)}$ be such that $B_{10^{-1}}(a_1^{(1)}) \cup \cdots \cup B_{10^{-1}}(a_{\ell_1}^{(1)}) \supseteq A$.
Let $a_{j_1}^{(1)}$ be such that $A \cap B_{10^{-1}}(a_{j_1}^{(1)})$ is not finitely covered by $\mathcal{S}$.
Let $a_1^{(2)}, \ldots, a_{\ell_2}^{(2)}$ be such that $B_{10^{-2}}(a_1^{(2)}) \cup \cdots \cup B_{10^{-2}}(a_{\ell_2}^{(2)}) \supseteq A \cap B_{10^{-1}}(a_{j_1}^{(1)})$.
Let $a_{j_2}^{(2)}$ be such that $A \cap B_{10^{-1}}(a_{j_1}^{(1)}) \cap B_{10^{-2}}(a_{j_2}^{(2)})$ is not finitely covered by $\mathcal{S}$.
Let $a_1^{(3)}, \ldots, a_{\ell_3}^{(3)}$ be such that $B_{10^{-3}}(a_1^{(3)}) \cup \cdots \cup B_{10^{-3}}(a_{\ell_3}^{(3)}) \supseteq A \cap B_{10^{-1}}(a_{j_1}^{(1)}) \cap B_{10^{-2}}(a_{j_2}^{(2)})$.
Let $a_{j_3}^{(3)}$ be such that $A \cap B_{10^{-1}}(a_{j_1}^{(1)}) \cap B_{10^{-2}}(a_{j_2}^{(2)}) \cap B_{10^{-3}}(a_{j_3}^{(3)})$ is not finitely covered by $\mathcal{S}$.
Continuing this process produces a sequence $(a_{j_1}^{(1)}, a_{j_2}^{(2)}, \ldots)$ which is Cauchy (If $m, n \geq k+1$ then

$d(a^{(m)}_{j_m}, a^{(n)}_{j_n}) \le d(a^{(m)}_{j_m}, a^{(k+1)}_{j_{k+1}}) + d(a^{(k+1)}_{j_{k+1}}, a^{(n)}_{j_n}) \le 10^{-(k+1)} + 10^{-(k+1)} \le 10^k)$.
Let $z \in A$.
To show: z is not a limit point of $(a^{(1)}_{j_1}, a^{(2)}_{j_2}, \ldots)$.
To show: There exists $\epsilon \in \mathbb{E}$ and $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\ge \ell}$ then $d(a^{(n)}_{j_n}, z) > \epsilon$.
Let $U \in \mathcal{S}$ such that $z \in U$.
Since U is open in X then there exists $k \in \mathbb{Z}_{>0}$ such that $B_{10^{-k}}(z) \subseteq U$.
Let $\epsilon = 10^{-k}$ and let $\ell = k$.
To show: If $n \in \mathbb{Z}_{\ge \ell}$ then $d(a^{(n)}_{j_n}, z) > \epsilon$.
Assume $n \in \mathbb{Z}_{\ge \ell}$.
Since $B_{10^{-n}}(a^{(n)}_{j_n}) \not\subseteq B_{10^{-k}}(z)$ there exists $y \in B_{10^{-n}}(a^{(n)}_{j_n})$ such that $d(y, z) > 10^{-k}$.
Thus $d(a^{(n)}_{j_n}, z) \ge d(y, z) - d(a^{(n)}_{j_n}, y) > 10^{-k} - 10^{-n} > 10^{-k} = \epsilon$.
So z is not a limit point of $(a^{(1)}_{j_1}, a^{(2)}_{j_2}, \ldots)$.
So A is not Cauchy compact. □

4.8.9 Subsets of $\mathbb{R}^n$

Proposition 4.22. *Let $\mathbb{R}^n$ have the standard metric and let $A \subseteq \mathbb{R}^n$.*

(a) (Bounded subsets of $\mathbb{R}^n$ are ball compact) If A is bounded then A is ball compact.

(b) (Closed subsets of $\mathbb{R}^n$ are Cauchy compact) If A is closed in $\mathbb{R}^n$ then A is Cauchy compact.

Proof.

(a) Assume $A \subseteq \mathbb{R}^n$ is bounded.
To show: A is ball compact.
To show: If $\epsilon \in \mathbb{E}$ then there exist $x_1, \ldots, x_\ell \in \mathbb{R}^n$ such that $A \subseteq B_\epsilon(x_1) \cup \cdots \cup B_\epsilon(x_\ell)$.
Since A is bounded then there exists $x \in \mathbb{R}_n$ and $M \in \mathbb{R}_{>0}$ such that $A \subseteq B_M(x)$.
Let $J = \{x + (c_1, \ldots, c_n) \in \mathbb{R}^n \mid c_i \in \{k10^{-\ell} \mid k \in \{-M, \ldots, M\}\}$.
Then

$$\Big(\bigcup_{y \in J} B_\epsilon(y)\Big) \supseteq B_M(x) \supseteq A \qquad \text{and} \qquad \mathrm{Card}(J) = (2M)^n.$$

So A is ball compact in $\mathbb{R}^n$. (EXACTLY WHAT PROPERTY OF $\mathbb{R}^n$ DID WE USE?? I THINK THIS IS THE ARCHIMEDEAN PROPERTY)

(b) Assume that A is closed in $\mathbb{R}^n$.
To show: A is Cauchy compact.
Since $\mathbb{R}^n$ is Cauchy compact and A is closed then, by Proposition 4.4(e), A is Cauchy compact. □

4.8.10 Equivalent characterizations of Hausdorff spaces

Theorem 4.23. *Let $(X, \mathcal{T})$ be a topological space. The following are equivalent.*

(H) If $x, y \in X$ and $x \neq y$ then there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V = \emptyset$.

(H1) If $x \in X$ then $\bigcap_{N \in \mathcal{N}(x)} \overline{N} = \{x\}$.

(H2) If $\Delta\colon X \to X \times X$ is the diagonal map then $\Delta(X)$ is closed in $X \times X$.

(H3) If I is a set and $\Delta\colon X \to \prod_{k \in I} X_k$, where $X_k = X$ for $k \in I$, is the diagonal map then $\Delta(X)$ is closed in $\prod_{k \in I} X_k$.

(H4) If $\mathcal{G}$ is a filter on X then $\mathcal{G}$ has at most one limit point.

(H5) If $\mathcal{J}$ is a filter on X and x is a limit point of $\mathcal{J}$ then x is the only cluster point of $\mathcal{J}$.

Proof.

(H3) $\Rightarrow$ (H2): (H2) is a special case of (H3).

(H2) $\Rightarrow$ (H): Assume $x, y \in X$ and $x \neq y$.

Then $(x, y) \in X \times X$ and $(x, y) \notin \Delta(X)$.
Thus, by (H2), $(x, y) \notin \overline{\Delta(X)}$.
So (x, y) is not a close point of $\Delta(X)$.
So there exists a neighborhood $Z \in \mathcal{N}((x, y))$ such that $Z \cap \Delta(X) = \emptyset$.
By the definition of the product topology,
there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $(U \times V) \cap \Delta(X) = \emptyset$.
So $U \cap V = \emptyset$.

(H) $\Rightarrow$ (H3):
Assume that if $x, y \in X$ and $x \neq y$ then there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V = \emptyset$.
To show: $\Delta(X)$ is closed in $\prod_{k \in I} X_k$, where $X_k = X$.

To show: If $x \in \prod_{k \in I} X_k$ and $x \notin \Delta(X)$ then x is not a close point of $\Delta(X)$.
Assume $x = (x_k) \in \prod_{k \in I} X_k$ and $x \notin \Delta(X)$.
To show: There exists $W \in \mathcal{N}(x)$ such that $W \cap \Delta(X) = \emptyset$.
Let $i, j \in I$ such that $x_i \neq x_j$.
Let $V_i \in \mathcal{N}(x_i)$ and $V_j \in \mathcal{N}(x_j)$ such that $V_i \cap V_j = \emptyset$.
Then $W = V_i \times V_j \times \prod_{k \neq i,j} X_k \in \mathcal{N}(x)$ and $W \cap \Delta(X) = \emptyset$.
So x is not a close point on $\Delta(X)$.
So $\Delta(X)$ is closed in $\prod_{k \in I} X_k$.

(H) $\Rightarrow$ (H1): Assume (H). Assume that if $x, y \in X$ and $x \neq y$ then there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V = \emptyset$.
To show: If $x \in X$ then $\bigcap_{N \in \mathcal{N}(x)} \overline{N} = \{x\}$.

Assume $x \in X$.
To show: If $y \in X$ and $y \notin \{x\}$ then $y \notin \bigcap_{N \in \mathcal{N}(x)} \overline{N}$.
Assume $y \in X$ and $y \notin \{x\}$.
To show: There exists $U \in \mathcal{N}(x)$ such that $y \notin \overline{U}$.
By (H), since $y \neq x$, there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V = \emptyset$.
So there exists $V \in \mathcal{N}(y)$ such that $V \cap U \neq \emptyset$.
So y is not a close point to U.
So $y \notin \overline{U}$.
So $y \notin \bigcap_{N \in \mathcal{N}(x)} \overline{N}$.

(H1) $\Rightarrow$ (H5): Assume that if $x \in X$ then $\bigcap_{N \in \mathcal{N}(x)} \overline{N} = \{x\}$.
To show: If $\mathcal{J}$ is a filter on X and x is a limit point of $\mathcal{J}$ then x is the only cluster point $\mathcal{J}$.

Assume $\mathcal{J}$ is a filter on X and x is a limit point of $\mathcal{J}$.
To show: If $y \in X$ is a cluster point of $\mathcal{J}$ then $y = x$.
Assume $y \in X$ is a cluster point of $\mathcal{J}$.
Since y is a cluster point of $\mathcal{J}$ then $y \in \bigcap_{M \in \mathcal{J}} \overline{M}$.
Since x is a limit point of $\mathcal{J}$ then $\mathcal{J} \supseteq \mathcal{N}(x)$.

So
$$y \in \Big(\bigcap_{M\in\mathcal{J}} \overline{M}\Big) \subseteq \Big(\bigcap_{N\in\mathcal{N}(x)} \overline{N}\Big) = \{x\}.$$

So $y = x$.

(H5) $\Rightarrow$ (H4): Assume that if $\mathcal{J}$ is a filter on X and x is a limit point of $\mathcal{J}$ then x is the only cluster point $\mathcal{J}$.
To show: If $\mathcal{G}$ is a filter on X then $\mathcal{G}$ has at most one limit point.

Assume $\mathcal{G}$ is a filter on X and x is a limit point of $\mathcal{G}$.
To show: If $y \in X$ is a limit point of $\mathcal{G}$ then $y = x$.
Assume $y \in X$ is a limit point of $\mathcal{G}$.
Since x is a limit point of $\mathcal{G}$ then $\mathcal{G} \supseteq \mathcal{N}(x)$.
So
$$x \in \Big(\bigcap_{N\in\mathcal{N}(x)} \overline{N}\Big) \supseteq \Big(\bigcap_{M\in\mathcal{G}} \overline{M}\Big).$$

So x is a cluster point of $\mathcal{G}$.
By (H5), y is the only cluster point of $\mathcal{G}$ and so $y = x$.

So $\mathcal{G}$ has at most one limit point.

(H4) $\Rightarrow$ (H): Assume not (H).

Let $x, y \in X$ with $x \neq y$ such that there do not exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ with $U \cap V = \emptyset$.
Let $\mathcal{J}$ be the filter generated by
$$\mathcal{B} = \{U \cap V \mid U \in \mathcal{N}(x), V \in \mathcal{N}(y)\}.$$

Since $X \in \mathcal{N}(y)$ then $\mathcal{N}(x) = \{U \cap X \mid U \in cN(x)\} \subseteq \mathcal{B} \subseteq \mathcal{J}$.
Since $X \in \mathcal{N}(x)$ then $\mathcal{N}(y) = \{X \cap V \mid V \in cN(y)\} \subseteq \mathcal{B} \subseteq \mathcal{J}$.
So x and y are both limit points of $\mathcal{J}$.
Since $x \neq y$ then (H4) does not hold.

$\square$

4.8.11 Equivalent characterizations of compact spaces

Theorem 4.24. *Let $(X, \mathcal{T})$ be a topological space. The following are equivalent.*

(C1) If $\mathcal{J}$ is an filter on X then there exists $x \in X$ such that x is a cluster point of $\mathcal{J}$.

(C2) If $\mathcal{G}$ is an ultrafilter on X then there exists $x \in X$ such that x is a limit point of $\mathcal{G}$.

(C3) If $\mathcal{C}$ is a collection of closed sets such that $\bigcap_{K\in\mathcal{C}} K = \emptyset$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $K_1, K_2, \ldots, K_\ell \in \mathcal{C}$ such that $K_1 \cap K_2 \cap \cdots \cap K_\ell = \emptyset$.

(C4) If $\mathcal{S}$ is a collection of open sets such that $\bigcap_{U\in\mathcal{S}} U = X$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{S}$ such that $U_1 \cup U_2 \cup \cdots \cup U_\ell = X$.

Proof. (Sketch)
(C3) $\Leftrightarrow$ (C4) by taking complements.
(C1) $\Rightarrow$ (C2): Assume (C1).

To show: If $\mathcal{G}$ is an ultrafilter on X then there exists $x \in X$ such that x is a limit point of $\mathcal{G}$.
Assume $\mathcal{G}$ is an ultrafilter on X.
By (C1), there exists $x \in X$ such that x is a cluster point of $\mathcal{G}$.
Since $\mathcal{G}$ is an ultrafilter x is a limit point of $\mathcal{G}$.

(C2) $\Rightarrow$ (C1): Assume (C2).

To show: If $\mathcal{J}$ is an filter on X then there exists $x \in X$ such that x is a cluster point of $\mathcal{J}$.
Assume $\mathcal{J}$ is a filter on X.
Since the collection of filters on X satisfies the hypotheses of Zorn's lemma, there exists an ultrafilter $\mathcal{G}$ such that $\mathcal{G} \supseteq \mathcal{J}$.
By (C2), there exists $x \in X$ such that x is a limit point of $\mathcal{G}$.
So x is a cluster point of $\mathcal{G}$.
Since $\mathcal{G} \supseteq \mathcal{J}$ and x is a cluster point of $\mathcal{G}$ then x is cluster point of $\mathcal{J}$.

(not C3) $\Rightarrow$ (not C1): Assume that there is a collection $\mathcal{C}$ of closed sets such that $\bigcap_{K \in \mathcal{C}} K = \emptyset$ but there does not exist $\ell \in \mathbb{Z}_{>0}$ and $K_1, K_2, \ldots, K_\ell \in \mathcal{C}$ such that $K_1 \cap K_2 \cap \cdots \cap K_\ell = \emptyset$.

Let $\mathcal{J}$ be the set of subsets of X which contain a set in $\mathcal{C}$.
Since there does not exist $\ell \in \mathbb{Z}_{>0}$ and $K_1, K_2, \ldots, K_\ell \in \mathcal{C}$ such that $K_1 \cap K_2 \cap \cdots \cap K_\ell = \emptyset$ the collection $\mathcal{J}$ is a filter.
Since $\bigcap_{N \in \mathcal{J}} \overline{N} \subseteq \bigcap_{K \in \mathcal{C}} \overline{K} = \bigcap_{K \in \mathcal{C}} K = \emptyset$, $\mathcal{J}$ does not have a cluster point.

(not C1) $\Rightarrow$ (not C3): Assume that there exists a filter $\mathcal{J}$ on X with no cluster point.

Then $\bigcap_{N \in \mathcal{J}} \overline{N} = \emptyset$.
Since $\mathcal{J}$ is a filter, if $\ell \in \mathbb{Z}_{>0}$ and $N_1, \ldots, N_\ell \in \mathcal{J}$ then $N_1 \cap \cdots \cap N_\ell \neq \emptyset$ and therefore $\overline{N_1} \cap \cdots \cap \overline{N_\ell} \neq \emptyset$.
Let $\mathcal{C} = \{\overline{N} \mid N \in \mathcal{J}\}$.
Then $\mathcal{C}$ is a collection of closed sets such that $\bigcap_{K \in \mathcal{C}} K = \emptyset$ but there does not exist $K_1, \ldots, K_\ell \in \mathcal{C}$ such that $K_1 \cap \cdots \cap K_\ell = \emptyset$.

□