MAST30026 Metric and Hilbert Spaces

Assignment 1

Due: 4pm Thursday August 11, 2022

Question 1. (Existence of eigenvectors)

(a) Let $\theta \in \mathbb{R}_{[0,2\pi)}$. Find the eigenvectors of $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ as a linear operator on $\mathbb{C}^2$.

(b) Show that $\begin{pmatrix} \cos(\frac{\pi}{4}) & \sin(\frac{\pi}{4}) \\ -\sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ does not have an eigenvector as a linear operator on $\mathbb{R}^2$.

(c) Let $n \in \mathbb{Z}_{>0}$ and let $A \in M_n(\mathbb{C})$. Prove carefully that A has an eigenvector as an operator on $\mathbb{C}^n$.

Question 2. (Radius of convergence) Let $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ be the $\mathbb{R}$-algebra with $i^2 = -1$ and $\overline{}\colon \mathbb{C} \to \mathbb{C}$ and $|\ |\colon \mathbb{C} \to \mathbb{R}_{\ge 0}$ given by

$$\overline{x+iy} = x - iy \qquad \text{and} \qquad |x+iy| = \sqrt{x^2+y^2}.$$

Let $\epsilon \in \mathbb{R}_{>0}$. Let $(a_1, a_2, \ldots)$ be a sequence in $\mathbb{C}$ and

$$\text{assume that} \qquad \sum_{n=1}^{\infty} a_n \epsilon^n \quad \text{exists in } \mathbb{C}.$$

Let $B_\epsilon(0) = \{z \in \mathbb{C} \mid |z| < \epsilon\}$. Prove carefully that

$$\text{if } z \in B_\epsilon(0) \quad \text{then} \quad \sum_{n=1}^{\infty} a_n z^n \quad \text{exists in } \mathbb{C}.$$

Question 3. (the dual of $\mathbb{R}^2$ in the $\|\ \|_p$ norm) If V is a normed $\mathbb{R}$-vector space with norm $\|\ \|_V$ and $\phi\colon V \to \mathbb{R}$ is a linear transformation then the *operator norm* of ϕ is

$$\|\phi\| = \sup\Big\{ \frac{\|\phi(v)\|_V}{\|v\|_V} \ \Big|\ v \in V \Big\}.$$

Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ be a linear functional. Let $a, b \in \mathbb{R}$ such that $\phi(x_1, x_2) = ax_1 + bx_2$. Prove carefully and directly that

(a) If $\mathbb{R}^2$ has norm given by $\|(x_1,x_2)\|_1 = |x_1| + |x_2|$ then

$$\|\phi\| = \max\{|a|, |b|\}.$$

(b) If $\mathbb{R}^2$ has norm given by $\|(x_1,x_2)\|_\infty = \max\{|x_1|, |x_2|\}$ then

$$\|\phi\| = |a| + |b|.$$

(c) If $p \in \mathbb{R}_{>1}$ and $\mathbb{R}^2$ has norm given by $\|(x_1, x_2)\|_p = (|x_1|^p + |x_2|^p)^{1/p}$ then

$$\|\phi\| = (|a|^q + |b|^q)^{1/q}, \qquad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Question 4. Let $\mathbb{R}^\infty = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R}\}$. For $x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$ define

$$\|x\|_1 = |x_1| + |x_2| + \cdots \qquad \text{and} \qquad \|x\|_\infty = \sup\{|x_1|, |x_2|, \ldots\}.$$

Define subspaces of $\mathbb{R}^\infty$ by

$$\begin{aligned}
c_c &= \{x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty \mid \text{all but a finite number of } x_i \text{ are } 0\}, \\
c_0 &= \big\{x = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \lim_{n\to\infty} x_n = 0\big\}, \\
\ell^1 &= \big\{x = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \|x\|_1 \text{ exists in } \mathbb{R}\big\}, \\
\ell^\infty &= \big\{x = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \|x\|_\infty \text{ exists in } \mathbb{R}\big\},
\end{aligned}$$

and for $p \in \mathbb{R}_{>1}$ define

$$\ell^p = \{x = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \|x\|_p \text{ exists in } \mathbb{R}\}, \quad \text{where} \qquad \|x\|_p = \Big(\sum_{i \in \mathbb{Z}_{>0}} |x_i|^p\Big)^{1/p}.$$

The purpose of this question is to establish and study the sequence

$$\begin{array}{ccccccccccccc}
c_c & \subsetneq & \ell^1 & \subsetneq & \ell^p & \subsetneq & \ell^2 & \subsetneq & \ell^q & \subsetneq & c_0 & \subsetneq & \ell^\infty \\
 & & \| & & \| & & \| & & \| & & & & \| \\
 & & (c_0)^* & \supsetneq & (\ell^q)^* & \supsetneq & (\ell^2)^* & \supsetneq & (\ell^p)^* & \supsetneq & & & (\ell^1)^*
\end{array}
\qquad \begin{array}{l} \text{for } p, q \in \mathbb{R}_{>1} \text{ with } p < 2 < q \\ \text{and } \ \frac{1}{p} + \frac{1}{q} = 1. \end{array}$$

Part A. (containment of vector spaces)

(a) Show that $c_c \subseteq c_0$ and $c_c \neq c_0$.

(b) Show that $c_c \subseteq \ell^1$ and $c_c \neq \ell^1$.

(c) Show that $\ell^1 \subseteq \ell^2$ and $\ell^1 \neq \ell^2$.

(d) Show that if $p \in \mathbb{R}_{>1}$ then $\ell^1 \subseteq \ell^p$ and $\ell^1 \neq \ell^p$.

(e) Show that if $p, q \in \mathbb{R}_{>1}$ and $p < q$ then $\ell^p \subseteq \ell^q$ and $\ell^p \neq \ell^q$.

(f) Show that if $q \in \mathbb{R}_{>1}$ then $\ell^q \subseteq c_0$ and $\ell^q \neq c_0$.

(g) Show that $c_0 \subseteq \ell^\infty$ and $c_0 \neq \ell^\infty$.

Part B. (the standard orthonormal sequence) Let W be a subspace of a normed vector space V. The *closure* of W is

$$\overline{W} = \Big\{ \lim_{n\to\infty} w_n \ \Big|\ (w_1, w_2, \ldots) \text{ is a sequence in } W \text{ and } \lim_{n\to\infty} w_n \text{ exists in } V \Big\}.$$

Let

$$e_1 = (1, 0, 0, 0, 0, \ldots), \quad e_2 = (0, 1, 0, 0, 0, \ldots), \quad e_3 = (0, 0, 1, 0, 0, \ldots), \quad \ldots,$$

(a) (the span) Show that $\mathrm{span}\{e_1, e_2, \ldots\} = c_c$.

(b) (the closure of the span in ℓ^p) Let $p \in \mathbb{R}_{>1}$. Show that, in ℓ^p, $\overline{\mathrm{span}\{e_1, e_2, \ldots\}} = \ell^p$.

(c) (the closure of the span in ℓ^1) Show that, in ℓ^1, $\overline{\mathrm{span}\{e_1, e_2, \ldots\}} = \ell^1$.

(d) (the closure of the span in ℓ^∞) Show that, in ℓ^∞, $\overline{\mathrm{span}\{e_1, e_2, \ldots\}} = c_0$.

Part D. (Duals) If V is a normed $\mathbb{R}$-vector space then

$$V^* = \{\phi\colon V \to \mathbb{R} \mid \phi \text{ is a linear transformation and } \|\phi\| \text{ exists in } \mathbb{R}\}.$$

(a) (Dual of c_0) Show that $\ell^1 = (c_0)^*$.

(b) (Dual of an ℓ^p-space) Let $p \in \mathbb{R}_{>1}$. Show that $\ell^q = (\ell^p)^*$, where $\frac{1}{p} + \frac{1}{q} = 1$.

(c) (Dual of ℓ^2) Show that $\ell^2 = (\ell^2)^*$.

(d) (Dual of ℓ^1) Show that $\ell^\infty = (\ell^1)^*$.

(e) (Dual of the dual of c_0) Show that $c_0 \subseteq ((c_0)^*)^*$ and $c_0 \neq ((c_0)^*)^*$.

(f) (Dual of the dual of ℓ^1) Show that $\ell^1 \subseteq ((\ell^1)^*)^*$ and $\ell^1 \neq ((\ell^1)^*)^*$.

Part E. (Completeness) Let V be a normed vector space with norm $\| \,\|_V$. The tolerance set is $\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}$. For $\epsilon \in \mathbb{E}$, the *ϵ-diagonal* is

$$B_\epsilon = \{(v, w) \in V \times V \mid \|v - w\|_V < \epsilon\}.$$

A *Cauchy sequence in* V is a sequence $(v_1, v_2, \ldots)$ such that

$$\text{if } \epsilon \in \mathbb{E} \text{ then there exists } N \in \mathbb{Z}_{>0} \text{ such that}$$

$$\text{if } m, n \in \mathbb{Z}_{\geq N} \text{ then } (v_m, v_n) \in B_\epsilon$$

(eventually the sequence is all inside the ϵ-diagonal). The normed vector space V is *complete* if it satisfies

$$\text{if } (v_1, v_2, \ldots) \text{ is a Cauchy sequence in } V \text{ then } \quad \lim_{n\to\infty} v_n \quad \text{exists in } V.$$

(a) (ℓ^1 is complete) Show that ℓ^1 is a complete normed vector space.

(b) (ℓ^p is complete) Let $p \in \mathbb{R}_{>1}$. Show that ℓ^p is a complete normed vector space.

(c) (ℓ^∞ is complete) Show that ℓ^∞ is a complete normed vector space.

Part F. (Completions) Let W be a subspace of a complete normed vector space V. The *completion* of W in V is

$$\widehat{W} = \left\{ \lim_{n\to\infty} w_n \;\middle|\; (w_1, w_2, \ldots) \text{ is a Cauchy sequence in } W \right\}.$$

(a) (The completion of c_c with respect to $\| \,\|_\infty$) Show that, in ℓ^∞, the completion of c_c is c_0.

(b) (The completion of c_c with respect to $\| \,\|_p$) Let $p \in \mathbb{R}_{>1}$. Show that, in ℓ^p, the completion of c_c is ℓ^p.