2.2 Assignment 2

2.2.1 Question 1: Sketch (selected steps skipped to focus on main points)

(a) Since $S = \{e_1, e_2, \ldots\}$ is a basis then $H = \overline{\text{span}\{e_1, e_2, \ldots\}}$ (note that here, basis means topological basis). Thus, by the construction of projection onto $W = \overline{\text{span}\{e_1, e_2, \ldots\}}$ for an orthonormal sequence $(e_1, e_2, \ldots)$, the projection onto W is the map $P\colon H \to H$ given by

$$P(x) = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n$$

(in particular, the limit of the partial sums exists in W).
By the orthogonal decomposition theorem, $H = W \oplus W^\perp$.
In this case $W = H$ and $W^\perp = H^\perp = 0$ (the last equality follows from the condition: if $v \in H$ and $\langle v, v\rangle = 0$ then $v = 0$).
So $x = P(x) + 0 \in H \oplus H^\perp$ and

$$x = P(x) = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n.$$

(c) Let $x, y \in H$. By part (a),

$$y = \sum_{n=1}^{\infty} \langle y, e_n\rangle e_n = \lim_{k\to\infty} s_k, \qquad \text{where} \quad s_k = \sum_{n=1}^{k} \langle y, e_n\rangle e_n.$$

Since $\langle , \rangle$ is continuous and $\lim_{k\to\infty} s_k$ exists in H and $\mathbb{R}_{\geq 0}$ is complete (this is a run on sentence and could be expanded to 2 or 3 separate steps) then $\lim_{k\to\infty}\langle x, s_k\rangle$ exists in $\mathbb{R}_{\geq 0}$ and

$$\begin{aligned}
\langle x, y\rangle &= \langle x, \lim_{k\to\infty} s_k\rangle = \lim_{k\to\infty} \langle x, s_k\rangle \\
&= \lim_{k\to\infty} \Big\langle x, \sum_{n=1}^{k} \langle y, e_n\rangle e_n \Big\rangle = \lim_{k\to\infty} \Big(\sum_{n=1}^{k} \langle x, e_n\rangle \overline{\langle y, e_n\rangle} \Big) \\
&= \sum_{n=1}^{\infty} \langle x, e_n\rangle \overline{\langle y, e_n\rangle}
\end{aligned}$$

(b) Using part (c),

$$\|x\|^2 = \langle x, x\rangle = \sum_{n=1}^{\infty} \langle x, e_n\rangle \overline{\langle x, e_n\rangle} = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2.$$

2.2.2 Question 2: computations

(a) The function $e_m(t)$ is an eigenvector of L with eigenvalue m since

$$Le_m(t) = \frac{d}{dt}(e^{imt}) = me^{imt} = me_m(t).$$

(b) Let $m, n \in \mathbb{Z}$ and assume $m \neq n$. Then

$$\begin{aligned}
\langle e_m(t), e_n(t)\rangle &= \frac{1}{2\pi} \int_0^{2\pi} e^{imt}\overline{e^{int}} dt = \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)t} dt \\
&= \frac{1}{2\pi} \Big(\frac{1}{i(m-n)} e^{i(m-n)t} \Big) \Big]_{t=0}^{t=2\pi} = \frac{1}{2\pi} \cdot \frac{1}{i(m-n)} (1-1) = 0.
\end{aligned}$$

Let $m, n \in \mathbb{Z}$ and assume $m = n$. Then

$$\langle e_n(t), e_n(t)\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{int}\overline{e^{int}}dt = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-n)t}dt = \frac{1}{2\pi}\int_0^{2\pi} dt = \frac{1}{2\pi}t\Big]_{t=0}^{t=2\pi} = \frac{2\pi}{2\pi} = 1.$$

So $(e_0, e_1, e_{-1}, e_2, e_{-2}, \ldots)$ is an orthonormal sequence in $L^2([0, 2\pi])$.
(c) If $n \in \mathbb{Z}_{\neq 0}$ then

$$\begin{aligned}\langle t, e_n(t)\rangle &= \frac{1}{2\pi}\int_0^{2\pi} te^{int}dt = \frac{1}{2\pi}t\frac{e^{int}}{in}\Big]_{t=0}^{t=2\pi} - \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{int}}{in}dt \\ &= \frac{1}{2\pi}\Big(\frac{2\pi}{in} - 0\Big) - \frac{1}{2\pi in}\frac{e^{int}}{in}\Big]_{t=0}^{t=2\pi} = \frac{1}{in} - \frac{1}{2\pi in}\Big(\frac{1}{in} - \frac{1}{in}\Big) = \frac{1}{in},\end{aligned}$$

and

$$\langle t, e_0(t)\rangle = \frac{1}{2\pi}\int_0^{2\pi} tdt = \frac{1}{2\pi}\frac{t^2}{2}\Big]_{t=0}^{t=2\pi} = \frac{1}{4\pi}(4\pi^2 - 0) = \pi,$$

then, by Question 1 part (a) (there there is a step skipped to show that $\overline{\mathrm{span}\{e_0, e_1, e_{-1}, \ldots\}} = L^2([0.2\pi])$, as with all steps it might not even be true, but if it is),

$$t = \pi + \sum_{n=1}^{\infty}\frac{1}{in}e^{int} + \frac{1}{-in}e^{-int} = \pi + \sum_{n=1}^{\infty}\frac{1}{in}(e^{int} - e^{-int}).$$

(d) Since

$$\begin{aligned}\langle t^2, e_n(t)\rangle &= \frac{1}{2\pi}\int_0^{2\pi} t^2e^{int}dt = \frac{1}{2\pi}t^2\frac{e^{int}}{in}\Big]_{t=0}^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} 2t\frac{e^{int}}{in}dt \\ &= \frac{1}{2\pi}\Big(4\pi^2\frac{1}{in} - 0\cdot\frac{1}{in}\Big) - \frac{1}{\pi in}\int_0^{2\pi} te^{int}dt = \frac{2\pi}{in} - \Big(\frac{1}{\pi in}t\frac{e^{int}}{in}\Big]_{t=0}^{t=2\pi}\Big) + \frac{1}{\pi in}\int_0^{2\pi}\frac{e^{int}}{in}dt \\ &= \frac{2\pi}{in} - \frac{1}{\pi in}\Big(\frac{2\pi}{in} - 0\Big) + \frac{-1}{\pi n^2}\frac{e^{int}}{in}\Big]_{t=0}^{t=2\pi} = \frac{2\pi}{in} + \frac{2}{n^2} - \frac{1}{\pi n^2}\Big(\frac{1}{in} - \frac{1}{in}\Big) = \frac{2\pi}{in} + \frac{2}{n^2},\end{aligned}$$

and

$$\langle t^2, e_0(t)\rangle = \frac{1}{2\pi}\int_0^{2\pi} t^2dt = \frac{1}{2\pi}\frac{t^3}{3}\Big]_{t=0}^{t=2\pi} = \frac{1}{6\pi}(8\pi^3 - 0) = \frac{4}{3}\pi^2,$$

then, by Question 1 part (a),

$$t^2 = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\Big(\frac{2\pi}{in} + \frac{2}{n^2}\Big)e^{int} + \Big(\frac{2\pi}{-in} + \frac{2}{n^2}\Big)e^{-int}.$$

2.2.3 Question 3: computations

(a) If $n \in \mathbb{Z}_{>0}$ then

$$\begin{aligned}e_n(t) &= e^{int} = \cos(nt) + i\sin(nt) = \frac{1}{\sqrt{2}}s_{-n}(t) + i\frac{1}{\sqrt{2}}s_n(t) \qquad \text{and} \\ e_{-n}(t) &= e^{-int} = \cos(nt) - i\sin(nt) = \frac{1}{\sqrt{2}}s_{-n}(t) - i\frac{1}{\sqrt{2}}s_n(t),\end{aligned}$$

and

$$s_n(t) = \frac{\sqrt{2}}{2i}(e_n(t) - e_{-n}(t)) \qquad \text{and} \qquad s_{-n}(t) = \frac{\sqrt{2}}{2}(e_n(t) + e_{-n}(t)).$$

If $m, n \in \mathbb{Z}_{>0}$ and $m \neq n$ then

$$\langle s_n(t), s_m(t)\rangle = \Big\langle \frac{\sqrt{2}}{2i}(e_n(t) - e_{-n}(t)), \frac{\sqrt{2}}{2i}(e_m(t) - e_{-m}(t))\Big\rangle = \frac{1}{2i\bar{i}}(0-0-0+0) = 0,$$

and

$$\langle s_n(t), s_{-m}(t)\rangle = \Big\langle \frac{\sqrt{2}}{2i}(e_n(t) - e_{-n}(t)), \frac{\sqrt{2}}{2}(e_m(t) + e_{-m}(t))\Big\rangle = \frac{1}{2i}(0+0-0-0) = 0.$$

Then

$$\langle s_n(t), s_n(t)\rangle = \Big\langle \frac{\sqrt{2}}{2i}(e_n(t) - e_{-n}(t)), \frac{\sqrt{2}}{2i}(e_n(t) - e_{-n}(t))\Big\rangle = \frac{1}{2i\bar{i}}(1-0-0+1) = 1,$$

and

$$\langle s_n(t), s_{-n}(t)\rangle = \Big\langle \frac{\sqrt{2}}{2i}(e_n(t) - e_{-n}(t)), \frac{\sqrt{2}}{2}(e_n(t) + e_{-n}(t))\Big\rangle = \frac{1}{2i}(1+0-0-1) = 0.$$

If $n \in \mathbb{Z}_{>0}$ then

$$\begin{aligned}
\langle s_0(t), s_n(t)\rangle &= \Big\langle e_0(t), \frac{\sqrt{2}}{2i}(e_n(t) - e_{-n}(t))\Big\rangle = 0 - 0 = 0, \\
\langle s_0(t), s_{-n}(t)\rangle &= \Big\langle e_0(t), \frac{\sqrt{2}}{2}(e_n(t) + e_{-n}(t))\Big\rangle = 0 - 0 = 0, \qquad \text{and} \\
\langle s_0(t), s_0(t)\rangle &= \langle e_0(t), e_0(t)\rangle = 1.
\end{aligned}$$

So $(s_0, s_1, s_{-1}, s_2, s_{-2}, \ldots)$ is an orthonormal sequence in $L^2(R_{[0,2\pi)})$.

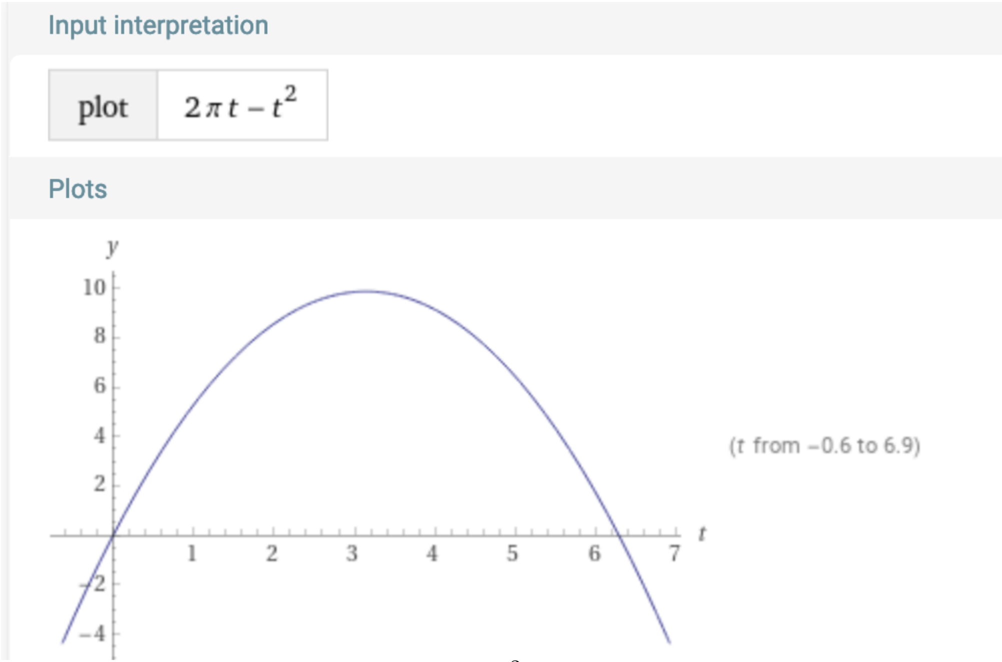
(b) Let $n \in \mathbb{Z}_{>0}$. Since $L = \frac{d^2}{dt^2}$ then

$$\begin{aligned}
Ls_n(t) &= \frac{d^2}{dt^2}(\sqrt{2}\sin(nt)) = \sqrt{2}\frac{d}{dt}n\cos(nt) = -\sqrt{2}n^2\sin(nt) = -n^2 s_n(t), \\
Ls_{-n}(t) &= \frac{d^2}{dt^2}(\sqrt{2}\cos(nt)) = -\sqrt{2}\frac{d}{dt}n\sin(nt) = -\sqrt{2}n^2\cos(nt) = -n^2 s_{-n}(t), \\
Ls_0(t) &= \frac{d^2}{dt^2}1 = 0 = 0s_0(t).
\end{aligned}$$

Thus, if $n \in \mathbb{Z}$ then the eigenvalue of L acting on $s_n(t)$ is $-n^2$.

(c) Since $f(t) = 2\pi t - t^2$ is a concave down parabola which goes through the points $(0,0)$ and $(0, 2\pi)$

the graph of $f(t)$ looks like



graph of $f(t) = 2\pi t - t^2$ from Wolfram alpha

This graph was obtained by a screenshot from Wolfram alpha by entering `plot 2pi*t-t^2`.

From Question 2 parts (d) and (e),

$$t = \pi + \sum_{n=1}^{\infty} \frac{1}{in}e^{int} + \frac{1}{-in}e^{-int} = \pi + \sum_{n=1}^{\infty} \frac{1}{in}(e^{int} - e^{-int})$$

and

$$t^2 = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{2\pi}{in}(e^{int} - e^{-int}) + \frac{2}{n^2}(e^{int} + e^{-int})$$

Thus (here there is a step skipped to show that $\overline{\mathrm{span}\{s_0, s_1, s_{-1}, \ldots\}} = L^2(\mathbb{R}_{[0.2\pi]})$, as with all steps it might not even be true, but if it is),

$$2\pi t - t^2 = \left(2\pi^2 - \frac{4}{3}\pi^2)\right) + \sum_{n=1}^{\infty} \frac{-2}{n^2}(e^{int} + e^{-int}) = \frac{2}{3}\pi^2 - \sum_{n=1}^{\infty} \frac{4}{n^2}\cos(nt).$$

Evaluating at $t = 2\pi$ gives

$$0 = \frac{2}{3}\pi^2 - 4\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{so that} \qquad \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

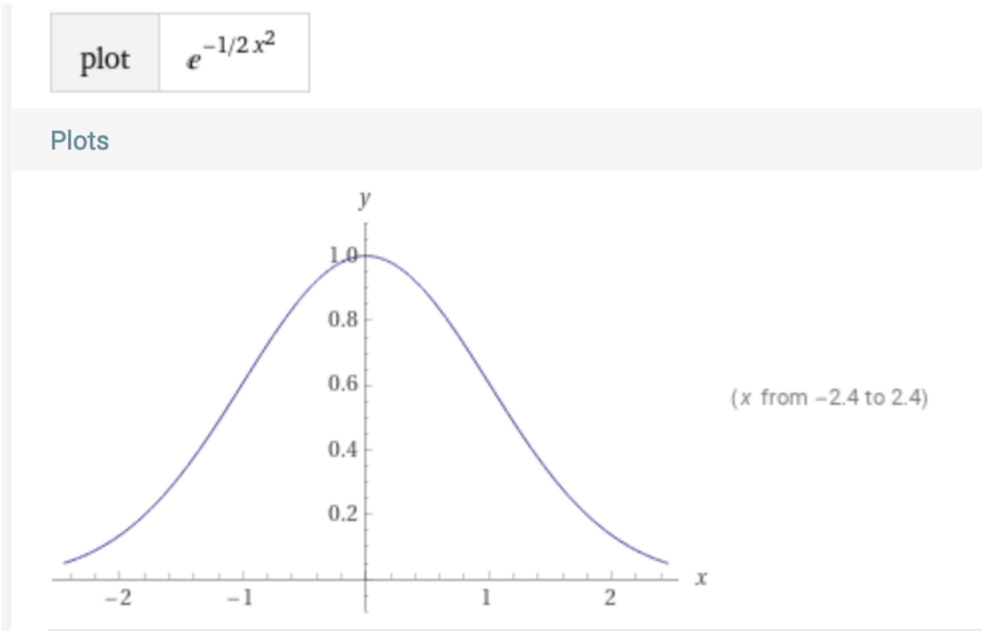
Evaluating at $t = \pi$ gives

$$\pi^2 = \frac{2}{3}\pi^2 - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \qquad \text{so that} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

2.2.4 Question 4: computations

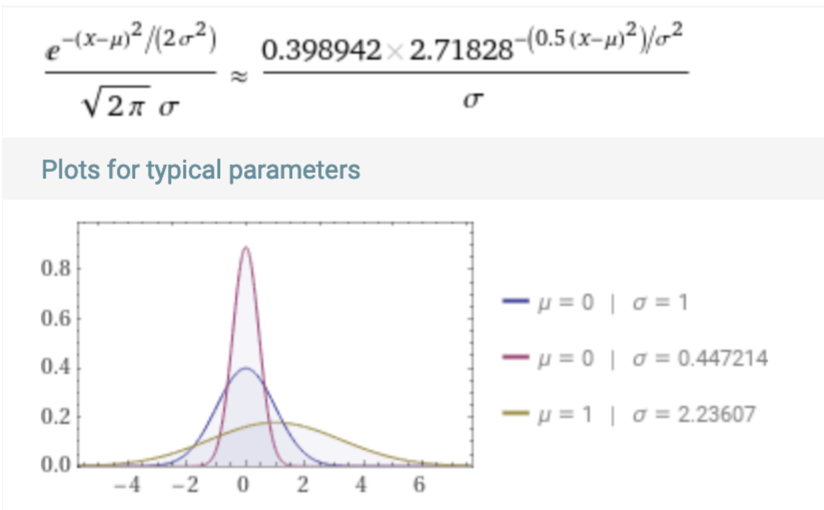
The graph of $N_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma}}$ is obtained from the graph of $w(x) = e^{-\frac{1}{2}x^2}$ by shifting and scaling (shift x by μ, scale the x-axis by σ^2 and scale the y-axis by $\sigma\sqrt{2\pi}$). The resulting graph is a bell curve symmetric about μ with standard deviation σ and with area under the curve equal to 1 so that it is the graph is the graph of a probability distribution.

Since the graph of $y = x^2$ is a parabola (symmetric about 0 and concave up) and the graph of $g = e^{-y}$ is decreasing to approach the line $g = 0$ then the graph of $w = e^{-\frac{1}{2}}x^2$ is a bell curve approaching $w = 0$ as $x \to \infty$ and $x \to -\infty$ and going through the point $(0, 1)$.



graph of $w(t) = e^{-\frac{1}{2}x^2}$ from Wolfram alpha

This graph was obtained by a screenshot from Wolfram alpha by entering `plot e^-(1/2)x^2` .



graph of $N_{\mu,\sigma}(x)$ from Wolfram alpha

This graph was obtained by a screenshot from Wolfram alpha by entering
`plot normal distribution mean mu standard deviation sigma`.

Every data analyst, statistician and probabilist must know these curves because of the central limit theorem, which says that the sum of a large number of independent variables will behave like a bell curve (see `https://en.wikipedia.org/wiki/Central_limit_theorem`).

Part (c): By definition, the Hermite polynomials $P_0, P_1, P_2, \ldots.$ are

$$P_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}\big(e^{-\frac{1}{2}x^2}\big)$$

Since

$$\begin{aligned}
\frac{d^0}{dx^0}(e^{-\frac{1}{2}x^2}) &= e^{-\frac{1}{2}x^2}, \\
\frac{d}{dx}(e^{-\frac{1}{2}x^2}) &= -xe^{-\frac{1}{2}x^2}, \\
\frac{d^2}{dx^2}(e^{-\frac{1}{2}x^2}) &= (-x)^2e^{-\frac{1}{2}x^2} - e^{-\frac{1}{2}x^2} = (x^2-1)e^{-\frac{1}{2}x^2}, \\
\frac{d^3}{dx^3}(e^{-\frac{1}{2}x^2}) &= (-x)(x^2-1)e^{-\frac{1}{2}x^2} + 2xe^{-\frac{1}{2}x^2} = (-x^3+3x)e^{-\frac{1}{2}x^2}, \\
\frac{d^4}{dx^4}(e^{-\frac{1}{2}x^2}) &= ((-x)(-x^3+3x) + (-3x^2+3))e^{-\frac{1}{2}x^2} = (x^4-6x^2+3)e^{-\frac{1}{2}x^2},
\end{aligned}$$

then

$$\begin{aligned}
P_0 &= 1, \\
P_1 &= x, \\
P_2 &= x^2-1, \\
P_3 &= x^3-3x, \\
P_4 &= x^4-6x^2+3.
\end{aligned}$$

Define operators $D\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$, $X\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$, $S\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ and $E\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ by

$$Df = \frac{df}{dx}, \qquad Xf = xf, \qquad Sf = e^{\frac{1}{2}x^2}f, \qquad \text{and} \qquad E = SDS^{-1}.$$

Then

$$E^n = SD^nS^{-1}, \qquad XD = DX - 1, \qquad SD = DS - XS, \qquad \text{and} \qquad SX = XS.$$

Hence $SDS^{-1} = D - X$. Then

$$DE^n = DSD^nS^{-1} = (SD + XS)D^nS^{-1} = SD^{n+1}S^{-1} + XSD^nS^{-1} = E^{n+1} + XE^n.$$

Since $P_n = (-1)^nE^n \cdot 1 = -E(-1)^{n-1}E^{n-1} \cdot 1 = -EP_{n-1}(x)$ then

$$\frac{d}{dx}P_n(x) = (-1)^nDE^n \cdot 1 = (-1)^n(E^{n+1} + XE^n) \cdot 1 = -P_{n+1}(x) + xP_n(x).$$

By induction,

$$XD^n = D^nX - nD^{n-1} \qquad \text{which gives} \qquad XE^n = E^nX - nE^{n-1},$$

since $XE^n = XSD^nS^{-1} = SXD^nS^{-1} = S(D^nX - nD^{n-1})S^{-1} = SD^nS^{-1}X - nSD^{n-1}S^{-1} = E^nX - nE^{n-1}$. Thus

$$\begin{aligned} xP_n(x) &= X(-1)^nE^n \cdot 1 = (-1)^n(E^nX - nE^{n-1}) \cdot 1 \\ &= (-1)^nE^nP_1(x) + nP_{n-1}(x) = P_{n+1}(x) + nP_{n-1}(x). \end{aligned}$$

So $P_{n+1}(x) = xP_n(x) - nP_{n-1}(x)$.

Since

$$\frac{d}{dx}P_n(x) = -P_{n+1}(x) + xP_n(x) = -P_{n+1}(x) + (P_{n+1}(x) + nP_{n-1}(x)) = nP_{n-1}(x).$$

Applying the operator identity $DX^n = X^nD + nX^{n-1}$ to the polynomial 1 gives

$$\frac{d}{dx}x^n = DX^n \cdot 1 = X^nD \cdot 1 + nX^{n-1} \cdot 1 = 0 + nx^{n-1} = nx^{n-1}.$$

(b) The favourite integral is

$$J = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\, dx.$$

then, putting $x = r\cos\theta$ and $y = r\sin\theta$ so that $r^2 = x^2 + y^2$,

$$\begin{aligned} J^2 &= \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}e^{-\frac{1}{2}y^2}\, dx\, dy = \int_{r=0}^{r=\infty}\int_{\theta=0}^{\theta=2\pi} e^{-\frac{1}{2}r^2} r dr\, d\theta \\ &= 2\pi \int_0^{\infty} re^{-\frac{1}{2}r^2} dr = -2\pi \int_0^{\infty} \big(-\tfrac{1}{2}2r\big)e^{-\frac{1}{2}r^2} dr \\ &= -2\pi \int_0^{-\infty} e^s ds = -2\pi e^s\Big]_{s=0}^{s=-\infty} = -2\pi(0-1) = 2\pi. \end{aligned}$$

Thus

$$J = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}.$$

A good reference is Exercise 51 of Chapter 2 of J. Rice, *Mathematical statistics and data analysis*, Duxbury Press 1995. This gives that

$$\langle P_0, P_0\rangle = \sqrt{2\pi}.$$

Using

$$P_n(x) = (-1)^n e^{\frac{1}{2}x^2}\frac{d^n}{dx^n}\big(e^{-\frac{1}{2}x^2}\big) \qquad \text{and} \qquad \langle f, g\rangle_w = \int_{-\infty}^{\infty} f(x)\overline{g(x)}e^{-\frac{1}{2}x^2}dx$$

then

$$\begin{aligned} (-1)^n\langle x^k, P_n(x)\rangle_w &= \int_{-\infty}^{\infty} (-1)^nx^kP_n(x)e^{-\frac{1}{2}x^2}dx = \int_{-\infty}^{\infty} x^k\frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2})dx \\ &= \int_{-\infty}^{\infty} x^k\frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2})dx \\ &= x^k\frac{d^{n-1}}{dx^{n-1}}(e^{-\frac{1}{2}x^2})\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} kx^{k-1}\frac{d^{n-1}}{dx^{n-1}}(e^{-\frac{1}{2}x^2})dx \\ &= x^k\frac{d^{n-1}}{dx^{n-1}}(e^{-\frac{1}{2}x^2})\Big]_{-\infty}^{\infty} - k(-1)^{n-1}\langle x^{k-1}, P_{n-1}(x)\rangle_w \\ &= x^kP_{n-1}(x)e^{-\frac{1}{2}x^2}\Big]_{-\infty}^{\infty} - 0 \\ &= \lim_{x\to\infty}\frac{x^kP_{n-1}(x)}{e^{\frac{1}{2}x^2}} - \lim_{x\to-\infty}\frac{x^kP_{n-1}(x)}{e^{\frac{1}{2}x^2}} = 0 - 0 = 0. \end{aligned}$$

Then

$$\begin{aligned}
\langle P_n(x), P_n(x)\rangle_w &= \int_{-\infty}^{\infty} P_n(x)P_n(x)e^{-\frac{1}{2}x^2}dx \\
&= \int_{-\infty}^{\infty} P_n(x)\frac{1}{n+1}\frac{d}{dx}\big(P_{n+1}\big)e^{-\frac{1}{2}x^2}dx \\
&= P_n(x)\frac{1}{n+1}P_{n+1}e^{-\frac{1}{2}x^2}\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}\big(P_n(x)e^{-\frac{1}{2}x^2}\big)\frac{1}{n+1}P_{n+1}dx \\
&= 0 - \int_{-\infty}^{\infty}\big(nP_{n-1}(x) - xP_n(x)\big)e^{-\frac{1}{2}x^2}\frac{1}{n+1}P_{n+1}(x)dx \\
&= \frac{n}{n+1}\langle P_{n-1}(x), P_{n+1}(x)\rangle_w + \frac{1}{n+1}\langle xP_n(x), P_{n+1}(x)\rangle_w \\
&= 0 + \frac{1}{n+1}\langle P_{n+1}(x) + nP_{n-1}(x), P_{n+1}(x)\rangle_w \\
&= \frac{1}{n+1}\langle P_{n+1}(x), P_{n+1}(x)\rangle_w.
\end{aligned}$$

Using the base case $\langle P_0(x), P_0(x)\rangle_w = \langle 1,1\rangle_w = \sqrt{2\pi}$ from part (b), then the induction step gives

$$\langle P_n(x), P_n(x)\rangle_w = n!\sqrt{2\pi}.$$

2.2.5 Question 5: computations

(a) Let $K = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}$ and $y = \left(\frac{2m\omega}{\hbar}\right)^{\frac{1}{2}}x$, then

$$h_r(x) = \frac{1}{\sqrt{r!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega}{2\hbar}x^2}P_r\Big(\big(\frac{2m\omega}{\hbar}\big)^{\frac{1}{2}}x\Big) = \frac{1}{\sqrt{r!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{1}{4}y^2}P_r(y),$$

and, using that $\langle P_r, P_s\rangle_w = \sqrt{2\pi}s!$ from Question 4 part (?),

$$\begin{aligned}
\langle h_r(x), h_s(x)\rangle &= \Big\langle \frac{1}{\sqrt{r!}}Ke^{-\frac{1}{2}y^2}P_r(y), \frac{1}{\sqrt{s!}}Ke^{-\frac{1}{2}y^2}P_s(y)\Big\rangle = \frac{1}{\sqrt{r!s!}}K^2\langle e^{-\frac{1}{4}y^2}P_r(y), e^{-\frac{1}{4}y^2}P_s(y)\rangle \\
&= \frac{1}{\sqrt{r!s!}}K^2\int_{-\infty}^{\infty} e^{-\frac{1}{4}y^2}P_r(y)\overline{e^{-\frac{1}{4}y^2}P_s(y)}dx \\
&= \frac{1}{\sqrt{r!s!}}K^2\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}P_r(y)P_s(y)\Big(\frac{\hbar}{2m\omega}\Big)^{\frac{1}{2}}dy = \frac{1}{\sqrt{r!s!}}\Big(\frac{m\omega}{\pi\hbar}\Big)^{\frac{1}{2}}\Big(\frac{\hbar}{2m\omega}\Big)^{\frac{1}{2}}\langle P_r, P_s\rangle_w \\
&= \begin{cases} \frac{1}{s!}\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\sqrt{2\pi}\,s!, & \text{if } r = s, \\ 0, & \text{if } r \neq s, \end{cases}
\end{aligned}$$

which gives that $\langle h_r, h_s\rangle = \delta_{rs}$.

(b)

$$a = \Big(\frac{m\omega}{2\hbar}\Big)^{\frac12}\big(x + i\frac{1}{m\omega}p\big) = \Big(\frac{m\omega}{2\hbar}\Big)^{\frac12}\big(x + i\frac{1}{m\omega}(-i\hbar)\frac{\partial}{\partial x}\big) = \Big(\frac{m\omega}{2\hbar}\Big)^{\frac12}\big(x + \frac{\hbar}{m\omega}(-i\hbar)\frac{d}{dx}\big),$$

$$a^\dagger = \Big(\frac{m\omega}{2\hbar}\Big)^{\frac12}\big(x - i\frac{1}{m\omega}p\big) = \Big(\frac{m\omega}{2\hbar}\Big)^{\frac12}\big(x - i\frac{1}{m\omega}(-i\hbar)\frac{\partial}{\partial x}\big) = \Big(\frac{m\omega}{2\hbar}\Big)^{\frac12}\big(x - \frac{\hbar}{m\omega}(-i\hbar)\frac{d}{dx}\big),$$

$$\begin{aligned} N = a^\dagger a &= \frac{m\omega}{2\hbar}\Big(x^2 - \frac{\hbar^2}{m^2\omega^2}\frac{d^2}{dx^2} - \frac{\hbar}{m\omega}\frac{d}{dx}x + \frac{\hbar}{m\omega}x\frac{d}{dx}\Big) \\ &= \frac{m\omega}{2\hbar}\Big(x^2 - \frac{\hbar^2}{m^2\omega^2}\frac{d^2}{dx^2} - \frac{\hbar}{m\omega}\big(x\frac{d}{dx}+1\big) + \frac{\hbar}{m\omega}x\frac{d}{dx}\Big) = \frac{m\omega}{2\hbar}\Big(x^2 - \frac{\hbar^2}{m^2\omega^2}\frac{d^2}{dx^2} - \frac{\hbar}{m\omega}\Big), \\ aa^\dagger &= \frac{m\omega}{2\hbar}\Big(x^2 - \frac{\hbar^2}{m^2\omega^2}\frac{d^2}{dx^2} + \frac{\hbar}{m\omega}\frac{d}{dx}x - \frac{\hbar}{m\omega}x\frac{d}{dx}\Big) \\ &= \frac{m\omega}{2\hbar}\Big(x^2 - \frac{\hbar^2}{m^2\omega^2}\frac{d^2}{dx^2} + \frac{\hbar}{m\omega}\big(x\frac{d}{dx}+1\big) - \frac{\hbar}{m\omega}x\frac{d}{dx}\Big) = \frac{m\omega}{2\hbar}\Big(x^2 - \frac{\hbar^2}{m^2\omega^2}\frac{d^2}{dx^2} + \frac{\hbar}{m\omega}\Big). \end{aligned}$$

So

$$aa^\dagger - a^\dagger a = \frac{m\omega}{2\hbar}\Big(\frac{\hbar}{m\omega} + \frac{\hbar}{m\omega}\Big) = 1.$$

Then

$$\begin{aligned} Na^\dagger - a^\dagger N &= a^\dagger a a^\dagger - a^\dagger a^\dagger a = a^\dagger(a^\dagger a + 1) - a^\dagger a^\dagger a = a^\dagger \quad \text{and} \\ Na - aN &= a^\dagger a a - a a^\dagger a = a^\dagger a a - (a^\dagger a + 1)a = -a. \end{aligned}$$

(c)

$$\begin{aligned} \hbar\omega(N + \tfrac12) &= \hbar\omega\Big(\frac{m\omega}{2\hbar}\big(x^2 - \frac{\hbar^2}{m^2\omega^2}\frac{d^2}{dx^2} - \frac{\hbar}{m\omega}\big) + \tfrac12\Big) = \tfrac12 m\omega^2x^2 - \frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \tfrac12\hbar\omega + \tfrac12\hbar\omega \\ &= \tfrac12 m\omega^2x^2 - \frac{\hbar^2}{2m}\frac{d^2}{dx^2} = \tfrac12 m\omega^2x^2 + \frac{1}{2m}(-i\hbar)^2\frac{d^2}{dx^2} = \tfrac12 m\omega^2x^2 + \frac{1}{2m}p^2 = H. \end{aligned}$$

That achieves the bulk of the marks for this assignment, we'll stop there.