

# MAST30026 Metric and Hilbert Spaces

## Assignment 2

Due: 4pm Thursday September 8, 2022

As usual, part of the exercise is to correct any typos and, in your solutions, to explain carefully to the marker what you corrected and why.

**Question 1. (Parseval's identities)** Let  $S = \{e_1, e_2, \dots\}$  be a countable orthonormal basis for a separable Hilbert space  $H$ . Prove that

(a)  $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$  (Fourier expansion),

(b)  $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$  (Parseval's identity for norms),

(c)  $\langle x, y \rangle = \sum_{n=1}^{\infty} \langle x, e_n \rangle \overline{\langle y, e_n \rangle}$  (Parseval's identity for inner products).

**Question 2 (Fourier decomposition for complex valued functions).** Let

$$L^2([0, 2\pi]) = \{f: [0, 2\pi] \rightarrow \mathbb{C} \mid \|f\|_2 \text{ exists in } \mathbb{R}_{\geq 0}\}$$

where the norm  $\|\cdot\|_2: L^2([0, 2\pi]) \rightarrow \mathbb{R}_{\geq 0}$  and the sesquilinear form  $\langle \cdot, \cdot \rangle: L^2([0, 2\pi]) \times L^2([0, 2\pi]) \rightarrow \mathbb{C}$  are given by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g(t)} dt \quad \text{and} \quad \|f\|_2 = \sqrt{\langle f, f \rangle}.$$

Define

$$e_m(t) = e^{imt}, \quad \text{for } m \in \mathbb{Z} \text{ and } t \in \mathbb{R}_{[0, 2\pi]}.$$

Let  $L: L^2(\mathbb{R}_{[0, 2\pi]}) \rightarrow L^2(\mathbb{R}_{[0, 2\pi]})$  be the operator on  $L^2(\mathbb{R}_{[0, 2\pi]})$  given by  $L = \frac{d}{dt}$ .

- (a) Let  $m \in \mathbb{Z}$ . Show that  $e_m$  is an eigenvector of  $L$  and compute the eigenvalue.
- (b) Show that  $(e_0, e_1, e_{-1}, e_2, e_{-2}, \dots)$  is an orthonormal sequence in  $L^2([0, 2\pi])$ .
- (c) Determine  $a_0, a_1, a_{-1}, a_2, a_{-2}, \dots \in \mathbb{C}$  such that  $t = a_0 e_0 + a_1 e_1 + a_{-1} e_{-1} + a_2 e_2 + a_{-2} e_{-2} + \dots$ .
- (d) Determine  $c_0, c_1, c_{-1}, c_2, c_{-2}, \dots \in \mathbb{C}$  such that  $t^2 = c_0 e_0 + c_1 e_1 + c_{-1} e_{-1} + c_2 e_2 + c_{-2} e_{-2} + \dots$ .

**Question 3 (Fourier decomposition for real valued functions).** Let

$$L^2([0, 2\pi]) = \{f: [0, 2\pi] \rightarrow \mathbb{R} \mid \|f\|_2 \text{ exists in } \mathbb{R}_{\geq 0}\}$$

where the norm  $\|\cdot\|_2: L^2([0, 2\pi]) \rightarrow \mathbb{R}_{\geq 0}$  and the sesquilinear form  $\langle \cdot, \cdot \rangle: L^2([0, 2\pi]) \times L^2([0, 2\pi]) \rightarrow \mathbb{R}$  are given by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x)g(x)dx \quad \text{and} \quad \|f\|_2 = \sqrt{\langle f, f \rangle}.$$

Let  $s_0(t) = 1$  for  $t \in \mathbb{R}_{[0, 2\pi]}$ . For  $n \in \mathbb{Z}_{>0}$  and  $t \in \mathbb{R}_{[0, 2\pi]}$  define

$$s_n(t) = \sqrt{2} \sin(nt), \quad \text{and} \quad s_{-n}(t) = \sqrt{2} \cos(nt).$$

- (a) Show that  $(s_0, s_1, s_{-1}, s_2, s_{-2}, \dots)$  form an orthonormal sequence in  $L^2([0, 2\pi])$ .  
 (b) Let  $L: L^2(\mathbb{R}_{[0, 2\pi]}) \rightarrow L^2(\mathbb{R}_{[0, 2\pi]})$  be the operator on  $L^2(\mathbb{R}_{[0, 2\pi]})$  given by

$$L = \frac{d^2}{dt^2}.$$

Let  $n \in \mathbb{Z}_{\geq 0}$ . Show that  $s_n$  and  $s_{-n}$  are eigenvectors of  $L$  and compute the eigenvalues.

- (c) Carefully graph the function  $f: \mathbb{R}_{[0, 2\pi]} \rightarrow \mathbb{R}$  given by  $f(t) = 2\pi t - t^2$ .  
 (d) Expand the function  $f: \mathbb{R}_{[0, 2\pi]} \rightarrow \mathbb{R}$  given by  $f(t) = 2\pi t - t^2$  in terms of the orthonormal sequence  $(s_0, s_1, s_{-1}, s_2, s_{-2}, \dots)$ .  
 (e) Carefully compute  $A, B \in \mathbb{C}$  such that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = A \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = B.$$

**Question 4 (Hermite polynomials).** Define a norm  $\| \cdot \|_w: \mathbb{C}[x] \rightarrow \mathbb{R}_{\geq 0}$  and a sesquilinear form  $\langle \cdot, \cdot \rangle: \mathbb{C}[x] \times \mathbb{C}[x] \rightarrow \mathbb{C}$  by

$$\langle f, g \rangle_w = \int_{-\infty}^{\infty} f(x) \overline{g(x)} e^{-\frac{1}{2}x^2} dx \quad \text{and} \quad \|f\|_w = \sqrt{\langle f, f \rangle_w}.$$

Define the Hermite polynomials  $P_n(x)$  by

$$P_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} (e^{-\frac{1}{2}x^2}), \quad \text{for } n \in \mathbb{Z}_{>0}.$$

Let  $\mu \in \mathbb{R}$  and  $\sigma \in \mathbb{R}_{>0}$ .

- (a) Carefully graph the “bell curve”  $w: \mathbb{R} \rightarrow \mathbb{R}$  and the “normal distribution”  $N_{\mu, \sigma}: \mathbb{R} \rightarrow \mathbb{R}$  which are given by

$$w(x) = e^{-\frac{1}{2}x^2} \quad \text{and} \quad N_{\mu, \sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

- (b) Carefully prove that

$$\langle 1, 1 \rangle_w = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1.$$

Explain why these integrals are vital to every data analyst, statistician and probabilist.

- (c) Define operators  $D: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]]$ ,  $X: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]]$ ,  $S: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]]$  and  $E: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]]$  by

$$Df = \frac{df}{dx}, \quad Xf = xf, \quad Sf = e^{\frac{1}{2}x^2} f, \quad \text{and} \quad E = SDS^{-1}.$$

Carefully prove the following identities:

$$E^n = SD^n S^{-1}, \quad XD = DX - 1, \quad SD = DS - XS, \quad SX = XS, \quad SDS^{-1} = D - X,$$

$$DE^n = E^{n+1} + XE^n, \quad XD^n = D^n X - nD^{n-1},$$

$$DX^n = X^n D + nX^{n-1}, \quad XE^n = E^n X - nE^{n-1}.$$

(c) Carefully compute  $P_0, P_1, P_2, P_3$  and  $P_4$  and prove that

$$\frac{d}{dx}P_n(x) = xP_n(x) - P_{n+1}(x), \quad P_{n+1}(x) = xP_n(x) - nP_{n-1}(x), \quad \text{and} \quad \frac{d}{dx}P_n(x) = nP_{n-1}(x).$$

(d) Show that if  $k, n \in \mathbb{Z}_{\geq 0}$  and  $k < n$  then  $\langle P_k, P_n \rangle_w = 0$ .

(e) Prove carefully that if  $n \in \mathbb{Z}_{\geq 0}$  then  $\langle P_n, P_n \rangle_w = \sqrt{2\pi} n!$ .

(f) Prove that  $(P_0, P_1, P_2, \dots)$  is an orthogonal basis of  $\mathbb{C}[x]$  with respect to  $\langle \cdot, \cdot \rangle_w$ .

**Question 5 (The quantum harmonic oscillator).** Let

$$L^2(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{C} \mid \|f\| \text{ exists in } \mathbb{R}_{\geq 0}\}$$

where the norm  $\|\cdot\|: L^2(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0}$  and the sesquilinear form  $\langle \cdot, \cdot \rangle: L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow \mathbb{C}$  are given by

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx \quad \text{and} \quad \|f\| = \sqrt{\langle f, f \rangle}.$$

Let  $m, \omega, \hbar \in \mathbb{R}_{>0}$  and define

$$h_r(x) = \frac{1}{\sqrt{r!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} P_r\left(\left(\frac{2m\omega}{\hbar}\right)^{1/2}x\right), \quad \text{for } r \in \mathbb{Z}_{\geq 0},$$

where  $P_r(x)$  are the Hermite polynomials of Question 4. Let  $i = \sqrt{-1}$  and define operators

$$p = -i\hbar \frac{d}{dx} \quad \text{and} \quad H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2x^2.$$

More explicitly, if  $f \in L^2(\mathbb{R})$  then  $Hf = \frac{-\hbar^2}{2m} \frac{d^2f}{dx^2} + \frac{1}{2}m\omega^2x^2f$ .

(a) Show that  $(h_0, h_1, h_2, \dots)$  is an orthonormal sequence in  $L^2(\mathbb{R})$ .

(b) Define operators  $a, a^\dagger$  and  $N$  on  $L^2(\mathbb{R})$  by

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2} \left(x + i\frac{1}{m\omega} \frac{\partial}{\partial x}\right), \quad a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2} \left(x - i\frac{1}{m\omega} \frac{\partial}{\partial x}\right) \quad \text{and} \quad N = aa^\dagger.$$

Prove that

$$N = \left(\frac{m\omega}{2\hbar}\right) \left(x^2 - \frac{\partial^2}{\partial x^2}\right), \quad aa^\dagger - a^\dagger a = 1, \quad Na^\dagger - a^\dagger N = a^\dagger, \quad Na - aN = -a.$$

(c) Show that, as operators on  $L^2(\mathbb{R})$ ,

$$H = \hbar\omega \left(N + \frac{1}{2}\right).$$

(d) Prove carefully that

$$a^\dagger h_n = (n+1)^{1/2} h_{n+1}, \quad ah_n = n^{1/2} h_{n-1}, \quad \text{and} \quad Nh_n = nh_n.$$

(e) Show that  $h_n$  is an eigenvector of  $H$  and compute the eigenvalue.