Question 5(a)

Let $\mathbb{C}((t)) = \left\{ a_k t^k + a_{k+1} t^{k+1} + \cdots \;\middle|\; \begin{matrix} k \in \mathbb{Z},\ a_k \neq 0 \\ a_i \in \mathbb{C} \end{matrix} \right\} \cup \{0\}$

Let $F = \left\{ \frac{f(t)}{g(t)} \;\middle|\; f(t), g(t) \in \mathbb{C}[[t]] \text{ and } g(t) \neq 0 \right\}$

To show: $F = \mathbb{C}((t))$.

To show: (aa) $\mathbb{C}((t)) \subseteq F$

(ab) $F \subseteq \mathbb{C}((t))$.

(aa) Let $p = a_k t^k + a_{k+1} t^{k+1} + \cdots \in \mathbb{C}((t))$.

Case 1: $k \geq 0$. Then $p \in \mathbb{C}[[t]]$ and

$$p = \frac{p}{1} \in F, \text{ where } 1 = 1 + 0t + 0t^2 + \cdots$$

Case 2: $k < 0$. Let $l = -k$ so that $l \in \mathbb{Z}_{>0}$ and

$$p = t^{-l}(a_k + a_{k+1} t + \cdots) = \frac{a_k + a_{k+1} t + \cdots}{t^l} \in F.$$

So $\mathbb{C}((t)) \subseteq F$.

(ab) To show: $F \subseteq \mathbb{C}((t))$.

Let $f = a_k t^k + a_{k+1} t^{k+1} + \cdots \in \mathbb{C}[[t]]$ and

$$g = b_l t^l + b_{l+1} t^{l+1} + \cdots \in \mathbb{C}[[t]]$$

so that $k, l \in \mathbb{Z}_{\geq 0}$ and $a_k \neq 0$ and $b_l \neq 0$.

Then $g = b_\ell t^\ell + b_{\ell+1} t^{\ell+1} + \cdots$

$$= b_\ell t^\ell (1 + b_\ell^{-1} b_{\ell+1} t + b_\ell^{-1} b_{\ell+2} t^2 + \cdots)$$

$$= b_\ell t^\ell (1 - (-b_\ell^{-1} b_{\ell+1} t - b_\ell^{-1} b_{\ell+2} t^2 - \cdots))$$

$$= b_\ell t^\ell (1-u), \quad \text{where}$$

$u = -b_\ell^{-1} b_{\ell+1} t - b_\ell^{-1} b_{\ell+2} t^2 - \cdots$

Since the lowest degree term of u^r is t^r or higher then

$$\frac{1}{1-u} = \sum_{r=0}^{\infty} u^r \text{ exists in } \mathbb{C}[[t]].$$

So
$$\frac{f}{g} = \frac{f}{b_\ell t^\ell (1-u)} = f b_\ell^{-1} t^{-\ell} \left(\sum_{r=0}^{\infty} u^r\right) \in \mathbb{C}((t)).$$

So $F \subseteq \mathbb{C}((t))$.

Thus $F = \mathbb{C}((t))$.

Question 5(b)

$d(a,b) = |b-a|$, where
$|a_k t^k + a_{k+1} t^{k+1} + \cdots| = 10^{-k}$ if $a_k \neq 0$.

To show: (a) If $x \in \mathbb{C}((t))$ then $d(x,x) = 0$,
(b) If $x, y \in \mathbb{C}((t))$ then $d(x,y) = d(y,x)$,
(c) If $x, y \in \mathbb{C}((t))$ and $d(x,y) = 0$ then $x = y$.
(d) If $x, y, z \in \mathbb{C}((t))$ then
$$d(x,y) \leq d(x,z) + d(z,y)$$

(a) Assume $x \in \mathbb{C}((t))$.
Then $x - x = 0$ and $d(x,x) = |x-x| = |0| = 0$.

(b) Assume $x, y \in \mathbb{C}((t))$. Then, if $x - y \neq 0$ then
$y - x = a_k t^k + a_{k+1} t^{k+1} + \cdots$ with $a_k \neq 0$ ~~then~~
and $x - y = -a_k t^k - a_{k+1} t^{k+1} + \cdots$ with $-a_k \neq 0$.
So $d(x,y) = 10^{-k} = d(y,x)$.

(c) Assume $x, y \in \mathbb{C}((t))$ and $d(x,y) = 0$.
Then $|y - x| = 0$ and so $y - x = 0 + 0t + \cdots$
So $y = x$.

(d) Assume $x, y, z \in \mathbb{C}((t))$. Let
$z - x = a_k t^k + a_{k+1} t^{k+1} + \cdots$ with $a_k \neq 0$
$y - z = b_\ell t^\ell + b_{\ell+1} t^{\ell+1} + \cdots$ with $b_\ell \neq 0$

then
$$y - x = a_k t^k + a_{k+1} t^{k+1} + \cdots$$
$$+ b_l t^l + b_{l+1} t^{l+1} + \cdots$$

So $d(x,y) = 10^{-\min(k,l)} \leq 10^{-k} + 10^{-l}$

$$= d(x,z) + d(z,y).$$

<u>Question 5(c)</u>

Let $\widehat{\mathbb{C}[t]}$ be the completion of $\mathbb{C}[t]$.

To show: $\mathbb{C}[[t]] \cong \widehat{\mathbb{C}[t]}$.

To show: There exists a bijective isometry
$\Phi : \widehat{\mathbb{C}[t]} \to \mathbb{C}[[t]]$.

Let $(p_1, p_2, \ldots) \in \widehat{\mathbb{C}[t]}$.

Since $(p_1, p_2, \ldots)$ is a Cauchy sequence there exists $k_N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{>k_N}$ then
$$|p_r - p_s| < 10^{-N}.$$

So the first N terms (powers t^k of t with $k \leq N$) of p_r and p_s are the same

Let $p = a_0 + a_1 t + \cdots$ be such that the coefficients $a_0, a_1, \ldots, a_N$ are the same as in p_r and set $\Phi(p_1, p_2, \ldots) = p$.

To define the inverse map

$\Psi : \mathbb{C}[[t]] \to \widehat{\mathbb{C}[t]}$ set

$\Psi(a_0 + a_1 t + \dots) = (a_0, a_0 + a_1 t, a_0 + a_1 t + a_2 t^2, \dots)$

Then $\Psi(a_0 + a_1 t + \dots) \in \widehat{\mathbb{C}[t]}$ and

$\Phi \circ \Psi = id$.

Finally $\hat{d}((p_1, p_2, \dots), (q_1, q_2, \dots)) = \lim\limits_{N \to \infty} d(p_N, q_N)$

and $d(p_N, q_N) = |q_N - p_N|$ as $\mathbb{C}[t] \subseteq \mathbb{C}[[t]]$.

So $\hat{d}(\Phi(p), \Phi(q)) = \lim\limits_{N \to \infty} |q_N - p_N|$

$= |q - p| = d(p, q)$

and Ψ is an isometry.

Question 5(d)

By part , we know that any Cauchy sequence in $\mathbb{C}[t]$ converges to an element of $\mathbb{C}[[t]]$. So any Cauchy sequence in $t^{-k}\mathbb{C}[t]$ converges to any element of $t^{-k}\mathbb{C}[[t]]$ (multiplication by t^{-k} is continuous.

So the completion of

$$R = \bigcup_{k=0}^{\infty} t^{-k}\mathbb{C}[t] \quad \text{is} \quad \bigcup_{k=0}^{\infty} \mathbb{C}[[t]] = \mathbb{C}((t)).$$

Since $R \subseteq \left\{ \frac{f(t)}{g(t)} \mid f(t), g(t) \in \mathbb{C}[t],\ g(t) \neq 0 \right\} = F$

then $\hat{R} \subseteq \hat{F}$. So $\mathbb{C}((t)) \subseteq \hat{F}$.

Since $\hat{R} = \mathbb{C}((t))$ then $\mathbb{C}((t))$ is complete.

So $\hat{F} \subseteq \mathbb{C}((t)) = \hat{R}$.

So $\hat{R} = \hat{F} = \mathbb{C}((t))$, where

$$F = \left\{ \frac{f(t)}{g(t)} \mid f(t), g(t) \in \mathbb{C}[t],\ g(t) \neq 0 \right\} = \mathbb{C}(t).$$

Question 5(e)

Let $k \in \mathbb{Z}_{>0}$. Let $z \in \mathbb{C}((t))$.

Case 1: $|z| = 10^{-k}$. Then $z = a_k t^k + a_{k+1} t^{k+1} + \cdots$

with $a_k \neq 0$ and

$$\frac{1}{\ell!} z^\ell = \frac{1}{\ell!} a_k^\ell t^{k\ell} + \text{higher terms}.$$

Let $s_n = 1 + z + \frac{1}{2!} z^2 + \cdots + \frac{1}{n!} z^n$ for $n \in \mathbb{Z}_{>0}$.

Then, if $m, n \in \mathbb{Z}_{>0}$ with $m < n$ then

$$|s_n - s_m| = \left| \frac{1}{(m+1)!} z^{m+1} + \cdots + \frac{1}{n!} z^n \right|$$
$$= \left| \frac{1}{(m+1)!} t^{k(m+1)} + \cdots \right| = 10^{-k(m+1)}$$

So $(s_1, s_2, s_3, \ldots)$ is a Cauchy sequence in $\mathbb{C}((t))$.
Since $\mathbb{C}((t))$ is complete this sequence converges.
We know $\mathbb{C}((t))$ is complete from part .

Case 2 $|z| = 10^k$. Then $z = a_{-k} t^{-k} + a_{-k+1} t^{-k+1} + \cdots$

with $a_{-k} \neq 0$ and

$$\frac{1}{\ell!} z^\ell = \frac{1}{\ell!} a_{-k}^\ell t^{-k\ell} + \text{higher terms}$$

Then

$$|s_n| = \left| 1 + z + \cdots + \frac{1}{n!} z^n \right| = \left| \frac{1}{n!} a_{-k}^n t^{-kn} + \text{higher terms} \right|$$
$$= 10^{kn}.$$

So $(s_1, s_2, s_3, \ldots)$ is an unbounded sequence in $\mathbb{C}((t))$.

So there does not exist $p \in \mathbb{C}((t))$ such that

$|s_n|$ gets closer and closer to $|p|$.

So $\sum_{r=0}^{\infty} \frac{1}{r!} z^r = (s_1, s_2, s_3, \ldots)$ does not converge in $\mathbb{C}((t))$.

Case 3 $|z| = 10^0 = 1$. Then $z = a_0 + a_1 t + \ldots$ with $a_0 \neq 0$.

If $z = 1 + 0t + 0t^2 + \cdots = 1$ then

$\exp(z) = 1 + 1 + \frac{1}{2!} + \cdots = e^1$ in $\mathbb{C}$.

So it looks like this sequence converges.
However, since the definition gives

$d(a,b) = |a-b| = 10^0 = 1$ for $a, b \in \mathbb{C}$

then the topology we are using on $\mathbb{C}$ is the discrete topology and this doesn't converge.