## Question 3 (a)



Let $x = 0_1$ and $y = 0_2$. Then $x \neq y$.

Let $U$ and $V$ be open such that

$x \in U$ and $y \in V$.

Then there exists $k, l \in \mathbb{Z}_{>0}$ with

$B_{10^{-k}}(0_1) \subseteq U$ and $B_{10^{-l}}(0_2) \subseteq V$.

Then let $m = \max\{k, l\}$. Then

$B_{10^{-m}}(0_1) \subseteq U$ and $B_{10^{-m}}(0_2) \subseteq V$.

Since

$10^{-(m+1)} \in B_{10^{-m}}(0_1)$ and $10^{-(m+1)} \in B_{10^{-m}}(0_2)$

then

$B_{10^{-m}}(0_1) \cap B_{10^{-m}}(0_2) \neq \emptyset$.

So $U \cap V \neq \emptyset$.

So there does not exist $U$ and $V$ open in $X$ with

$0_1 \in V$ and $0_2 \in V$ and $U \cap V = \emptyset$.

So $X$ is not Hausdorff.

Question 3(b) MH5Ass3 Q3b ①

Let $E = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Let $z \in X$.

Case 1: $z = 0_1$.

Let $N \in \mathcal{N}(z)$. Then there exists $k \in \mathbb{Z}_{>0}$ with $B_{10^{-k}}(0_1) \subseteq N$.

Then
$$\frac{1}{10^{k+1}} \in B_{10^{-k}}(0) \subseteq N.$$

So $E \cap N \neq \emptyset$.

So $0_1$ is a close point to $E$.

Case 2: $z = 0_2$.

Let $N \in \mathcal{N}(z)$. Then there exists $k \in \mathbb{Z}_{>0}$ with $B_{10^{-k}}(0_2) \subseteq N$.

Then
$$\frac{1}{10^{k+1}} \in B_{10^{-k}}(0_2) \subseteq N.$$

So $E \cap N \neq \emptyset$.

So $0_2$ is a close point to $E$.

Case 3: $z \in E$.

Then $z$ is a close point to $E$.

Case 4: $z \notin E$ and $z \neq 0_1$ and $z \neq 0_2$ and $z < 1$.

Let $n \in \mathbb{Z}_{>0}$ be such that
$$\frac{1}{n} < z < \frac{1}{n-1}.$$

Then $B_{10^{-n}}(z) \in \mathcal{N}(z)$ and $B_{10^{-n}}(z) \cap E = \emptyset$.

So $z$ is not a close point to $E$.

Case 5: $z \notin E$ and $z \neq 0_1$ and $z \neq 0_2$ and $z > 1$.

Let $k \in \mathbb{Z}_{>0}$ such that $\frac{1}{10^k} < \frac{z-1}{2}$.

Then $B_{10^{-k}}(z) \cap E = \emptyset$ and $B_{10^{-k}}(z) \in \mathcal{N}(z)$.

So $z$ is not a close point to $E$.

Thus

$$\overline{\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}} = \{\text{close points to } E\}$$
$$= \{0_1, 0_2\} \cup E$$
$$= \{0_1, 0_2, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}.$$