Assignment 3 Question 2

Let $\mathcal{C} = \{C \subseteq \mathbb{R} \mid C \text{ is finite}\} \cup \{\emptyset, \mathbb{R}\}$.

(a) $\mathcal{T}$ will be a topology on $\mathbb{R}$ if $\mathcal{C}$ satisfies

(aa) $\emptyset \in \mathcal{C}$ and $\mathbb{R} \in \mathcal{C}$

(ab) If $\mathcal{S} \subseteq \mathcal{C}$ then $\left(\bigcap_{C \in \mathcal{S}} C\right) \in \mathcal{C}$

(ac) If $k \in \mathbb{Z}_{>0}$ and $C_1, C_2, \ldots, C_k \in \mathcal{C}$ then $C_1 \cup \cdots \cup C_k \in \mathcal{C}$.

(aa) follows by definition of $\mathcal{C}$.

(ab) follows since intersections of finite sets are finite or empty.

(ac) follows since the union of a finite number of finite sets is finite.

(b) To show: $(\mathbb{R}, \mathcal{T})$ is not Hausdorff.

To show: There exist $x, y \in \mathbb{R}$ with $x \neq y$ such that there does not exist $U, V \in \mathcal{T}$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

To show: There exist $x, y \in \mathbb{R}$ with $x \neq y$ such that there do not exist

$C, D \in \mathcal{C}$ with $x \in C^c$, $y \in D^c$ and

$C \cup D = \mathbb{R}$.

Let $x = 1$ and $y = 2$.

To show: There do not exist $C, D \in \mathcal{C}$ with $x \in C^c$, $y \in D^c$ and $C \cup D = \mathbb{R}$.

Case 1: $C, D \in \mathcal{C}$ are finite sets and $C \neq \mathbb{R}$ and $D \neq \mathbb{R}$. Then $C \cup D \neq \mathbb{R}$.

Case 2: $C = \mathbb{R}$ and $D \in \mathcal{C}$ is finite and $D \neq \mathbb{R}$. Then $C^c = \emptyset$ and $x \notin C^c$.

Case 3 $C \in \mathcal{C}$ is finite and $D = \mathbb{R}$ and $C \neq \mathbb{R}$. Then $D^c = \emptyset$ and $y \notin D^c$.

Case 4 $C = \mathbb{R}$ and $D = \mathbb{R}$. Then $x \notin C^c$.

So there do not exist $C, D \in \mathcal{C}$ with $x \in C^c$, $y \in D^c$ and $C \cup D = \mathbb{R}$.

So $(X, \mathcal{T})$ is not Hausdorff.

(c) $\overline{\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}} \in \mathcal{C}$ since $\overline{\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}}$ is closed.

Since $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \subseteq \overline{\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}}$ then

$\overline{\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}}$ is not finite and not empty.

So $\overline{\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}} = \mathbb{R}$.

So every point of $\mathbb{R}$ is a limit point of the sequence $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$ in $(\mathbb{R}, \mathcal{T})$.

Question 2(d)

Let $E \subseteq \mathbb{R}$. Let $\mathcal{T}$ be the Zariski topology on $\mathbb{R}$. Let

$$A, B \in \mathcal{T} \text{ with } A \neq \emptyset \text{ and } B \neq \emptyset.$$

Let $F_A = A^c$ and $F_B = B^c$ so that F_A and F_B are finite and

$$\mathbb{R} = A \sqcup F_A \quad \text{and} \quad \mathbb{R} = B \sqcup F_B.$$

If $p \notin A \cup B$ then $p \notin A$ and $p \notin B$ and so $p \in F_A$ and $p \in F_B$. So

$$\mathbb{R} = (A \cup B) \sqcup (F_A \cap F_B)$$

If $p \notin A \cap B$ then $p \notin A$ or $p \notin B$ and so $p \in F_A$ or $p \in F_B$. So

$$\mathbb{R} = (A \cap B) \sqcup (F_A \cup F_B)$$

Since $\mathbb{R} = (A \cup B) \sqcup (F_A \cap F_B)$ then

$$E = (E \cap (A \cup B)) \sqcup (E \cap F_A \cap F_B)$$

So

$E \cap (A \cup B) = E$ if and only if $E \cap F_A \cap F_B = \emptyset$.

Since $\mathbb{R} = (A \cap B) \cup (F_A \cup F_B)$ then

$$E = (E \cap A \cap B) \cup (E \cap (F_A \cup F_B)).$$

So

$E \cap A \cap B = \phi$ if and only if $E \cap (F_A \cup F_B) = E$.

Thus, if $A, B \in \mathcal{J}$ form a disconnection of E (i.e. $E \cap A \neq \phi$, $E \cap B \neq \phi$, $E \cap (A \cup B) = E$, $E \cap A \cap B = \phi$)

then $E \cap (F_A \cup F_B) = E$ and E is finite and, since $E \cap A \neq \phi$ and $E \cap B \neq \phi$ and $(E \cap A) \cap (E \cap B) = \phi$, E contains at least two points.

So, if E is connected then E is infinite or E is a single point.

Indeed, if $p_1, p_2 \in E$ with $p_1 \neq p_2$ and E is finite then

if $A \in \mathcal{J}$ is infinite containing p_1 and

$B \in \mathcal{J}$ is infinite containing the other points of E

then $A, B \in \mathcal{J}$ form a disconnection of E.

So the connected sets of $\mathbb{R}$ are the infinite subsets of $\mathbb{R}$ and the one point subsets of $\mathbb{R}$.

Question 2(c)

MHSRss3Q2c. (1)

Let $E \subseteq \mathbb{R}$.

To show E is compact.

Let $\mathcal{S}$ be an open cover of E so that

$$E \subseteq \left(\bigcup_{u \in \mathcal{S}} u\right).$$

Let $V \in \mathcal{S}$.

Case 1: $V \supseteq \mathbb{R}$. Then $E \subseteq V$ and the single set $\{V\}$ is a finite subcover of E.

Case 2 $V \not\supseteq \mathbb{R}$. Then V^c is finite since V is open.

So there exists $\ell \in \mathbb{Z}_{>0}$ with

$$(V^c \cap E) = \{e_1, \ldots, e_\ell\}.$$

If $i \in \{1, \ldots, \ell\}$ then there exists $V_i \in \mathcal{S}$ with $e_i \in V_i$ since $\mathcal{S}$ covers E.

Then

$$V \cup V_1 \cup \cdots \cup V_\ell \supseteq E.$$

So $\mathcal{S}$ has a finite subcover.

So E is compact. //

Question 2(f)

By part (b), $(\mathbb{R}, \mathcal{T})$ is not Hausdorff.
By the following proposition a metric space (with the metric space topology) is Hausdorff. So there does not exist a metric $d: \mathbb{R}\times\mathbb{R} \to \mathbb{R}_{\geq 0}$ such that the metric space topology for d is equal to $\mathcal{T}$.

Proposition Let (X, d) be a metric space and let $\mathcal{T}_d$ be the metric space topology. Then $(X, \mathcal{T}_d)$ is Hausdorff.

Proof Let $x, y \in X$ with $x \neq y$.

Let $k \in \mathbb{Z}_{>0}$ such that $10^{-k} < \frac{d(x,y)}{2}$.

Then $B_{10^{-k}}(x)$ is open in $\mathcal{T}_d$ and $x \in B_{10^{-k}}(x)$
and $B_{10^{-k}}(y)$ is open in $\mathcal{T}_d$ and $y \in B_{10^{-k}}(y)$.

To show: $B_{10^{-k}}(x) \cap B_{10^{-k}}(y) = \emptyset$.

Let $z \in B_{10^{-k}}(x)$

To show: $d(z,y) \geq 10^{-k}$.

Since $d(x,y) \leq d(x,z) + d(z,y) < 10^{-k} + d(z,y)$

then
$$d(z,y) > d(x,y) - 10^{-k} > d(x,y) - \frac{d(x,y)}{2}$$
$$= \frac{d(x,y)}{2} > 10^{-k}.$$

So $z \notin B_{10^{-k}}(y)$.

So $B_{10^{-k}}(x) \cap B_{10^{-k}}(y) = \emptyset$.

So $(X, \mathcal{T}_d)$ is Hausdorff. //.