Assignment 3 Question 1

(a) To show: $d: (X \times Y) \times (X \times Y) \to \mathbb{R}_{\geq 0}$ given by
$$d((x_1,y_1),(x_2,y_2)) = d_X(x_1,x_2) + d_Y(y_1,y_2)$$
is a metric on $X \times Y$.

To show: (aa) If $(x,y) \in X \times Y$ then $d((x,y),(x,y)) = 0$.

(ab) If $(x_1,y_1),(x_2,y_2) \in X \times Y$ and
$$d((x_1,y_1),(x_2,y_2)) = 0 \text{ then } (x_1,y_1) = (x_2,y_2).$$

(ac) If $(x_1,y_1),(x_2,y_2) \in X \times Y$ then
$$d((x_1,y_1),(x_2,y_2)) = d((x_2,y_2),(x_1,y_1)).$$

(ad) If $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in X \times Y$ then
$$d((x_1,y_1),(x_2,y_2)) \leq d((x_1,y_1),(x_3,y_3)) + d((x_2,y_2),(x_3,y_3)).$$

(ad) Assume $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in X \times Y$.

Then
$$\begin{aligned} d((x_1,y_1),(x_2,y_2)) &= d_X(x_1,x_2) + d_Y(y_1,y_2) \\ &\leq d_X(x_1,x_3) + d_X(x_3,x_2) + d_Y(y_1,y_3) + d_Y(y_3,y_2) \\ &= (d_X(x_1,x_3) + d_Y(y_1,y_3)) + (d_X(x_3,x_2) + d_Y(y_3,y_2)) \\ &= d((x_1,y_1),(x_3,y_3)) + d((x_3,y_3),(x_2,y_2)). \end{aligned}$$

(aa) Assume $(x,y) \in X \times Y$.

To show: $d((x,y),(x,y)) = 0$.

$$d((x,y),(x,y)) = d_X(x,x) + d_Y(y,y)$$
$$= 0 + 0 = 0.$$

(ab) Assume $(x_1,y_1),(x_2,y_2) \in X \times Y$. and

~~To show~~: $d((x_1,y_1),(x_2,y_2)) = 0$.

To show: $(x_1,y_1) = (x_2,y_2)$.

To show: $x_1 = x_2$ and $y_1 = y_2$.

$$0 = d((x_1,y_1),(x_2,y_2)) = d_X(x_1,x_2) + d_Y(y_1,y_2)$$

Since $d_X(x_1,x_2) \in \mathbb{R}_{\geq 0}$ and $d_Y(y_1,y_2) \in \mathbb{R}_{\geq 0}$

then $d_X(x_1,x_2) = 0$ and $d_Y(y_1,y_2) = 0$.

So $x_1 = x_2$ and $y_1 = y_2$.

(ac) Assume $(x_1,y_1),(x_2,y_2) \in X \times Y$.

To show: $d((x_1,y_1),(x_2,y_2)) = d((x_2,y_2),(x_1,y_1))$.

By symmetry of d_X and d_Y, ~~commutativity of addition in $\mathbb{R}_{\geq 0}$~~,

$$d((x_1,y_1),(x_2,y_2)) = d_X(x_1,x_2) + d_Y(y_1,y_2)$$
$$= d_X(x_2,x_1) + d_Y(y_2,y_1)$$
$$= d((x_2,y_2),(x_1,y_1)).$$

Question 1(b)

To show: $\mathcal{T}_m = \mathcal{T}$ where

$\mathcal{T}_m$ is generated by $\{B_\varepsilon(x,y) \mid \varepsilon \in \mathbb{E}, (x,y) \in X \times Y\}$.

$\mathcal{T}$ is generad by $\{B_{\varepsilon_1}(x) \times B_{\varepsilon_2}(y) \mid \varepsilon_1, \varepsilon_2 \in \mathbb{E}, x \in X, y \in Y\}$.

To show (ba) $\mathcal{T} \subseteq \mathcal{T}_m$

(bb) $\mathcal{T}_m \subseteq \mathcal{T}$.

(ba) To show: If $U \in \mathcal{T}$ then $U \in \mathcal{T}_m$.

Assume $U \in \mathcal{T}$.

To show: $U \in \mathcal{T}_m$.

To show: If $(x,y) \in U$ then there exists $\varepsilon \in \mathbb{E}$ such that $B_\varepsilon(x,y) \subseteq U$.

Assume $(x,y) \in U$.

Since (x,y) is an interior point of U there exist $\delta_1, \delta_2 \in \mathbb{E}$ with $B_{\delta_1}(x) \times B_{\delta_2}(y) \subseteq U$

To show: There exists $\varepsilon \in \mathbb{E}$ such that $B_\varepsilon(x,y) \subseteq U$.

To show: There exists $\varepsilon \in \mathbb{E}$ such that $B_\varepsilon(x,y) \subseteq B_{\delta_1}(x) \times B_{\delta_2}(y)$

Let $\cancel{\varepsilon = \delta_1 + \delta_2}$ $\varepsilon = \min(\delta_1, \delta_2)$

Let $(z,w) \in B_\varepsilon(x,y)$. Then $\cancel{d_X(z,x) \le d_X(z,x) + d_Y(w,y)}$

$\cancel{d((z,w),(x,y)) = d_X(z,x) + d_Y(w,y) \le}$ $\cancel{= d((z,w),(x,y))}$

then
$\delta_1 \geq \varepsilon > d((z,w),(x,y)) \geq d_X(z,x) + d_X(w,y) \geq d_X(z,x)$
and
$\delta_2 \geq \varepsilon > d((z,w),(x,y)) = d_X(z,x) + d_X(w,y) \geq d_Y(w,y)$.

So $(z,w) \in B_{\delta_1}(x) \times B_{\delta_2}(y)$.

So $(z,w) \in \mathcal{U}$.

So $B_\varepsilon(x,y) \subseteq \mathcal{U}$.

So $\mathcal{U} \in \mathcal{T}_m$.

(b) To show: $\mathcal{T}_m \subseteq \mathcal{T}$.

To show: If $\mathcal{U} \in \mathcal{T}_m$ then $\mathcal{U} \in \mathcal{T}$

Assume $\mathcal{U} \in \mathcal{T}_m$

To show: $\mathcal{U} \in \mathcal{T}$.

To show: If $(x,y) \in \mathcal{U}$ then there exists $\varepsilon_1 \in \mathbb{E}$, $\varepsilon_2 \in \mathbb{E}$ with $B_{\varepsilon_1}(x) \times B_{\varepsilon_2}(y) \subseteq \mathcal{U}$

Assume $(x,y) \in \mathcal{U}$.

Since (x,y) is an interior point of $\mathcal{U}$ then there exists $\varepsilon \in \mathbb{E}$ with $B_\varepsilon(x,y) \subseteq \mathcal{U}$.

To show: There exists $\varepsilon_1 \in \mathbb{E}$, $\varepsilon_2 \in \mathbb{E}$ with $B_{\varepsilon_1}(x) \times B_{\varepsilon_2}(y) \subseteq \mathcal{U}$.

To show: There exists $\varepsilon_1 \in \mathbb{E}$, $\varepsilon_2 \in \mathbb{E}$ with $B_{\varepsilon_1}(x) \times B_{\varepsilon_2}(y) \subseteq B_\varepsilon(x,y)$.

Let $\varepsilon_1 = \frac{1}{10}\delta$ and $\varepsilon_2 = \frac{1}{10}\delta$

To show: $B_{\varepsilon_1}(x) \times B_{\varepsilon_2}(y) \subseteq B_\delta(x,y)$

Let $(z,w) \in B_{\varepsilon_1}(x) \times B_{\varepsilon_2}(y)$.

To show: $(z,w) \in B_\delta(x,y)$

To show: $d((z,w),(x,y)) < \delta$.

$d((z,w),(x,y)) \leq d_X(z,x) + d_Y(z,w) < \varepsilon_1 + \varepsilon_2$.

$= \frac{1}{10}\delta + \frac{1}{10}\delta < \delta$

So $(z,w) \in B_\delta(x,y) \subseteq \mathcal{U}$.

So $B_{\varepsilon_1}(x) \times B_{\varepsilon_2}(y) \subseteq \mathcal{U}$.

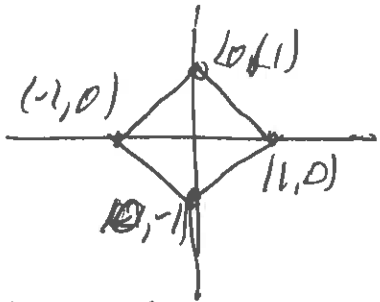
So $\mathcal{U} \in \mathcal{T}$.

So $\mathcal{T}_m \subseteq \mathcal{T}$.

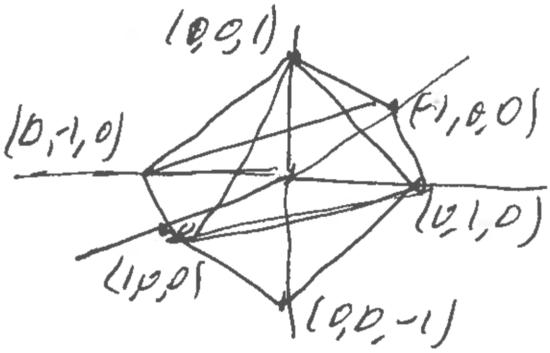
So $\mathcal{T}_m = \mathcal{T}$. //.

Question 1(c)

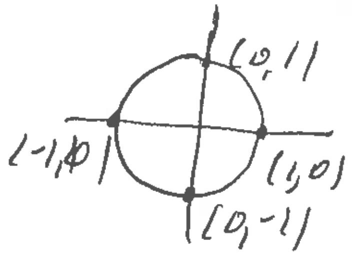
A sketch of $\{(x_1,x_2)\in\mathbb{R}^2 \mid |x_1|+|x_2|=1\}$ is



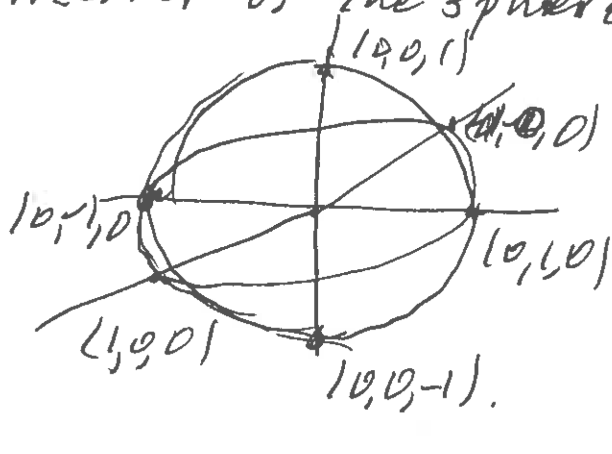
A sketch of $\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid |x_1|+|x_2|+|x_3|<1\} = B_1^{d_1}(0)$ is the interior of the octahedron



A sketch of $\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1^2+x_2^2=1\}$ is

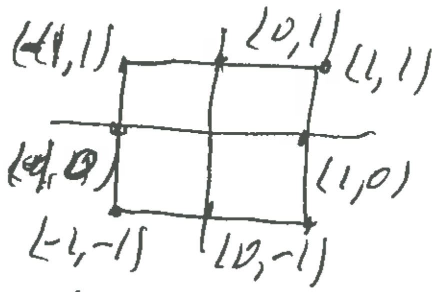


A sketch of $\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid \sqrt{x_1^2+x_2^2+x_3^2}<1\} = B_1^{d_2}(0)$ is the interior of the sphere

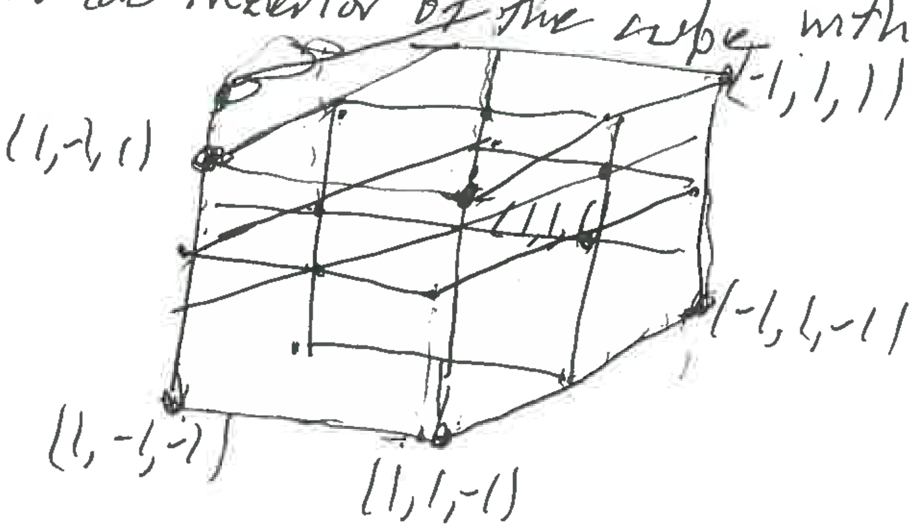


MHJ Ass 3 Q1c (2)

A sketch of $\{(x_1,x_2)\in\mathbb{R}^2 \mid \max\{|x_1|,|x_2|\}\leq 1\}$ is



A sketch of $\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid \max\{|x_1|,|x_2|,|x_3|\} < 1\} = B_1^3(0)$ is the interior of the cube with vertices in the set



$\{(\pm1,\pm1,\pm1)\}$.