Assignment 1 Question 4E

We'd like to show that ℓ^1, ℓ^p and ℓ^∞ are complete by using Question 4F.

The same argument as for question 4F using part (c) of Question 4B (which says $\hat{\varepsilon}_c = \ell^1$ with respect to the $\| \|_1$ norm) will show that

$$\hat{\varepsilon}_c = \ell^1, \text{ with respect to the } \ell^1\text{-norm}.$$

So Question 4F gives that

$$\ell^1 = \hat{\varepsilon}_c \text{ with norm } \| \|_1,$$
$$\ell^p = \hat{\varepsilon}_c \text{ with norm } \| \|_p,$$
$$\ell^\infty = \hat{\varepsilon}_c \text{ with norm } \| \|_\infty.$$

So if we know that the completion is complete then ℓ^1, ℓ^p and ℓ^∞ are complete finishing Question 4E.

We know that the completion is complete by part (b) of the proof of Theorem 5.2 in the Completions notes on the course webpage. This proof is copied on the next pages.

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Let $(\hat{X}, \hat{d})$ be the completion of (X, d).

To show: $(\hat{X}, \hat{d})$ is complete

To show: If $\vec{x}_1, \vec{x}_2, \ldots$ is a Cauchy sequence in $\hat{X}$ then $\vec{x}_1, \vec{x}_2, \ldots$ converges.

Assume
$$\begin{aligned}\vec{x}_1 &= (x_{11}, x_{12}, x_{13}, \ldots)\\ \vec{x}_2 &= (x_{21}, x_{22}, x_{23}, \ldots)\\ \vec{x}_3 &= (x_{31}, x_{32}, x_{33}, \ldots)\\ &\vdots\end{aligned}$$
is a Cauchy sequence in $\hat{X}$.

To show: There exists $\vec{z} = (z_1, z_2, \ldots)$ in $\hat{X}$ such that $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.

Using that $\overline{\varphi(X)} = \hat{X}$,

for $k \in \mathbb{Z}_{>0}$ let $z_k \in X$ such that $\hat{d}(\varphi(z_k), \vec{x}_k) < \frac{1}{10^k}$

$\vec{x}_k = (x_{k1}, x_{k2}, x_{k3}, \ldots)$, $\varphi(z_k) = (z_k, z_k, z_k, \ldots)$

and $\hat{d}(\varphi(z_k), \vec{x}_k) < 10^{-k}$

To show: (a) $\vec{z} = (z_1, z_2, \ldots) \in \hat{X}$

(b) $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.

(a) To show: $\vec{z} = (z_1, z_2, \ldots)$ is a Cauchy sequence in X

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To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $r,s \in \mathbb{Z}_{\geq \ell}$ then $d(z_r, z_s) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: there exists $\ell \in \mathbb{Z}_{>0}$ such that if $r,s \in \mathbb{Z}_{\geq \ell}$ then $d(z_r, z_s) < \varepsilon$.

Let $\ell_1 = \frac{10}{\varepsilon} + 1$ so that $\frac{1}{\ell_1} < \frac{\varepsilon}{10}$.

Let $\ell_2 \in \mathbb{Z}_{>0}$ be such that if $r,s \in \mathbb{Z}_{\geq 0}$ then $\hat{d}(\vec{x}_r, \vec{x}_s) < \frac{\varepsilon}{10}$.

Let $\ell = \max(\ell_1, \ell_2)$.

To show: If $r,s \in \mathbb{Z}_{\geq \ell}$ then $d(z_r, z_s) < \varepsilon$.

Assume $r,s \in \mathbb{Z}_{\geq \ell}$

To show: $d(z_r, z_s) < \varepsilon$.

$$\begin{aligned} d(z_r, z_s) &= \hat{d}(\varphi(z_r), \varphi(z_s)) \\ &\leq \hat{d}(\varphi(z_r), \vec{x}_r) + d(\vec{x}_r, \vec{x}_s) + \hat{d}(\vec{x}_s, \varphi(z_s)) \\ &\leq 10^{-r} + \frac{\varepsilon}{10} + 10^{-s} \leq \frac{1}{\ell_1} + \frac{\varepsilon}{10} + \frac{1}{\ell_1} \\ &< \frac{\varepsilon}{10} + \frac{\varepsilon}{10} + \frac{\varepsilon}{10} < \varepsilon. \end{aligned}$$

So z is a Cauchy sequence.

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(b) To show: $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.

To show: $\lim_{n\to\infty} \hat{d}(\vec{x}_n, \vec{z}) = 0$

$$\lim_{n\to\infty} \hat{d}(\vec{x}_n, \vec{z}) \le \lim_{n\to\infty} \left(\hat{d}(\vec{x}_n, \varphi(z_n)) + \hat{d}(\varphi(z_n), \vec{z})\right)$$

$$\le \lim_{n\to\infty} 10^{-n} + \hat{d}(\varphi(z_n), \vec{z})$$

$$= \lim_{n\to\infty} 10^{-n} + \lim_{n\to\infty} \hat{d}(\varphi(z_n), \vec{z})$$

$$= 0 + 0 = 0.$$

So $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.

So the Cauchy sequence $(\vec{x}_1, \vec{x}_2, \ldots)$ in $\hat{X}$ converges in $\hat{X}$.

So $(\hat{X}, \hat{d})$ is a complete metric space.