Assignment 1 Question 4D

(c) is a special case of (b) with

$2 = p$ and $2 = q$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{p} + \frac{1}{q}$.

(e) By part (a), $c_0^* = \ell^1$.

So we need to show $c_0 \subseteq (\ell^1)^*$ and $c_0 \neq (\ell^1)^*$.

Use the map

$$\begin{aligned} \Phi: c_0 &\longrightarrow (\ell^1)^* \\ x &\longmapsto \varphi_x : \ell^1 \longrightarrow \mathbb{R} \\ &\qquad\quad w \longmapsto \langle w, x \rangle \end{aligned}$$

where $\langle w, x\rangle = \sum_{k=1}^{\infty} w_k x_k$ if $w = (w_1, w_2, \ldots)$
and $x = (x_1, x_2, \ldots)$.

To show: (a) If $x \in c_0$ and $w \in \ell^1$ then $\langle w, x\rangle$ exists in $\mathbb{R}$.

(b) Φ is an isometry.

From (b) we can conclude that Φ is injective and it will make sense to write $c_0 \subseteq (\ell^1)^*$.

Then we will need to show:

(c) There exists $\varphi \in (\ell^1)^*$ such that $\varphi \notin \operatorname{im} \Phi$ in order to conclude that Φ is not a bijection.
Then it will make sense to write $c_0 \neq (\ell^1)^*$.

(b) Let $p \in \mathbb{R}_{>1}$. To show: $\ell^q \cong (\ell^p)^*$ where $\frac{1}{p}+\frac{1}{q}=1$.

The point of this question is to work through and understand the proof of Theorem 14.2 in the Notes on Function Spaces on the web page for the course. Then we might expect that parts (a) and (d) (showing that $\ell^1 \cong (c_0)^*$ and $\ell^\infty \cong (\ell^1)^*$) might be done similarly.

Here is a handwritten copy of the proof of Theorem 14.2.

Define
$$\begin{aligned} \Psi: \ell^q &\longrightarrow (\ell^p)^* \\ y &\longmapsto \varphi_y: \ell^p \to \mathbb{R} \\ &\qquad\quad x \longmapsto \langle y, x\rangle. \end{aligned}$$

where $\langle y, x\rangle = \sum_{k=1}^{\infty} y_k x_k$ if $y=(y_1,y_2,\ldots)$ and $x=(x_1,x_2,\ldots)$.

To show: (ba) φ is a linear transformation.

(bb) φ is injective

(bc) φ is surjective

(bd) If $y \in \ell^q$ then $\|\varphi_y\| = \|y\|$.

(ba) To show: (baa) If $y_1, y_2 \in \ell^q$ then $\varphi(y_1+y_2) = \varphi(y_1)+\varphi(y_2)$
(bab) If $y \in \ell^q$ and $c \in \mathbb{R}$ then $\varphi(cy) = c\varphi(y)$.

(baa) Assume $y_1, y_2 \in \ell^q$

To show: $\Phi(y_1+y_2) = \Phi(y_1)+\Phi(y_2)$

To show: If $x \in \ell^p$ then $\varphi_{y_1+y_2}(x) = (\varphi_{y_1}+\varphi_{y_2})(x)$.

Assume $x \in \ell^p$.

To show: $\varphi_{y_1+y_2}(x) = (\varphi_{y_1}+\varphi_{y_2})(x)$

$$\begin{aligned}\varphi_{y_1+y_2}(x) &= \langle y_1+y_2, x\rangle = \langle y_1, x\rangle + \langle y_2, x\rangle \\ &= \varphi_{y_1}(x)+\varphi_{y_2}(x) = (\varphi_{y_1}+\varphi_{y_2})(x).\end{aligned}$$

(bab) Assume $y \in \ell^q$ and $c \in \mathbb{R}$

To show: $\Phi(cy) = c\Phi(y)$.

To show: If $x \in \ell^p$ then $\varphi_{cy}(x) = (c\varphi_y)(x)$.

Assume $x \in \ell^p$

To show: $\varphi_{cy}(x) = (c\varphi_y)(x)$.

$$\varphi_{cy}(x) = \langle cy, x\rangle = c\langle y, x\rangle = c(\varphi_y(x)) = (c\varphi_y)(x).$$

So Φ is a linear transformation.

Note that is part of the proof did not use anything about p and q so it will also work when $p=1$ or $p=\infty$.

(bb) To show: $\Psi : \ell^q \to (\ell^p)^*$ is invertible.

To show: There exists $\Phi : (\ell^p)^* \to \ell^q$ such that $\Phi \circ \Psi = \mathrm{id}$ and $\Psi \circ \Phi = \mathrm{id}$.

Define
$$\Phi : (\ell^p)^* \to \ell^q \text{ by}$$
$$\Phi(\varphi) = (\varphi(e_1), \varphi(e_2), \ldots) \text{ where } e_i = (0, \ldots, 0, 1, 0, \ldots)$$
with 1 in the i^{th} spot.

To show: (bba) $\Psi \circ \Phi = \mathrm{id}$

(bbb) $\Phi \circ \Psi = \mathrm{id}$

(bba) To show: If $\varphi \in (\ell^p)^*$ then $\Psi(\Phi(\varphi)) = \varphi$.

Assume $\varphi \in (\ell^p)^*$.

To show: $\Psi(\Phi(\varphi)) = \varphi$.

To show: If $x \in \ell^p$ then $\Psi(\Phi(\varphi))(x) = \varphi(x)$.

Assume $x \in \ell^p$ with $x = (x_1, x_2, \ldots)$.

To show: $\Psi(\Phi(\varphi))(x) = \varphi(x)$.

$$\Psi(\Phi(\varphi))(x) = \Psi(\varphi(e_1), \varphi(e_2), \ldots)(x)$$
$$= \langle (\varphi(e_1), \varphi(e_2), \ldots), (x_1, x_2, \ldots) \rangle$$
$$= \sum_{k=1}^{\infty} \varphi(e_k) x_k = \varphi\left(\sum_{k=1}^{\infty} x_i e_i \right) = \varphi(x). \qquad (*)$$

Note that the line (*) will probably not work if $p=\infty$ since $\sum_{k=1}^{\infty} x_i e_i \in \overline{\mathrm{span}\{e_1, e_2, \ldots\}} = c_0$ and $c_0 \neq \ell^\infty$.

(bbb) To show: $\Phi \circ \Psi = id$.

To show: If $y \in \ell^q$ then $\Phi(\Psi(y)) = y$.

Assume $y \in \ell^q$ with $y = (y_1, y_2, \ldots)$.

To show: $(\Phi \circ \Psi)(y) = y$.

$$\Phi(\Psi(y)) = \Phi(\varphi_y) = (\varphi_y(e_1), \varphi_y(e_2), \ldots)$$
$$= (y_1, y_2, \ldots) = y.$$

(bc) To show: If $y \in \ell^q$ then $\|\varphi_y\| = \|y\|_q$

Assume $y \in \ell^q$ with $y = (y_1, y_2, \ldots)$.

To show: (bca) $\|\varphi_y\| \leq \|y\|_q$

(bcb) $\|\varphi_y\| \geq \|y\|_q$

(bca) To show: If $x \in \ell^p$ then $|\varphi_y(x)| \leq \|x\|_p \cdot \|y\|_q$

Assume $x \in \ell^p$ with $x = (x_1, x_2, \ldots)$.

Then by Hölder's inequality,

$$|\varphi_y(x)| = |\langle y, x \rangle| = \left|\sum_{n=1}^{\infty} x_n y_n\right| \leq \|x\|_p \|y\|_q.$$

So $\dfrac{|\varphi_y(x)|}{\|x\|_p} \le \|y\|_q$.

So $\|\varphi_y\| = \sup\left\{ \dfrac{|\varphi_y(x)|}{\|x\|_p} \,\Big|\, x \in \ell^p, x \ne 0 \right\} \le \|y\|_q$.

So $\|\varphi_y\| \le \|y\|_q$.

(bcb) To show: $\|\varphi_y\| \ge \|y\|_q$

To show: There exists $x \in \ell^p$ with $\dfrac{|\varphi_y(x)|}{\|x\|_p} \ge \|y\|_q$.

Let $x = (\mathrm{sgn}(y_1)|y_1|^{q-1}, \mathrm{sgn}(y_2)|y_2|^{q-1}, \ldots)$

Then
$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} = \left(\sum_{n=1}^{\infty} |\mathrm{sgn}(y_n)|y_n|^{q-1}|^p\right)^{1/p}$$
$$= \left(\sum_{n=1}^{\infty} |y_n|^{pq-p}\right)^{1/p} = \left(\sum_{n=1}^{\infty} |y_n|^{pq(1-\frac{1}{q})}\right)^{1/p}$$
$$= \left(\sum_{n=1}^{\infty} |y_n|^{pq\frac{1}{p}}\right)^{1/p} = \left(\left(\sum_{n=1}^{\infty} |y_n|^q\right)^{\frac{1}{q}}\right)^{q\frac{1}{p}}$$
$$= \|y\|_q^{q\frac{1}{p}} = \|y\|_q^{q(1-\frac{1}{q})} = \|y\|_q^{q-1}$$

So
$$|\varphi_y(x)| = \left|\sum_{n=1}^{\infty} x_n y_n\right| = \left|\sum_{n=1}^{\infty} (\mathrm{sgn}(y_n)|y_1|)(\mathrm{sgn}(y_n)|y_n|^{q-1})\right|$$
$$= \sum_{n=1}^{\infty} |y_n|^q = \|y\|_q^q = \|y\|_q \|y\|_q^{q-1} = \|y\|_q \|x\|_p.$$

So $\|\varphi_y\| \ge \|y\|_q$.

The same proof of local would work nicely when $p=1$ and $q=\infty$ if we know that if $x \in \ell^1$ and $y \in \ell^\infty$ then (7)

MHSAss1(4)
A. Ram

$$|\langle x,y\rangle| \le \|x\|_1 \|y\|_\infty$$

Proof Let $x=(x_1,x_2,\dots)\in \ell^1$ so that $\sum_{n=1}^{\infty} |x_n| = \|x\|_1$ exists in $\mathbb{R}_{\ge 0}$.

Let $y=(y_1,y_2,\dots)\in \ell^\infty$ so that

$$\|y\|_\infty = \sup\{|y_1|,|y_2|,\dots\} \text{ exists in } \mathbb{R}_{\ge 0}.$$

Then $\langle x,y\rangle = \sum_{n=1}^{\infty} x_n y_n$ converges in $\mathbb{R}$

since

$$\sum_{n=1}^{\infty} |x_n||y_n| \le \|y\|_\infty \cdot \sum_{n=1}^{\infty} |x_n| = \|y\|_\infty \|x\|_1$$

so that $\sum_{n=1}^{\infty} |x_n y_n|$ converges in $\mathbb{R}_{\ge 0}$.

Then

$$|\langle x,y\rangle| = \left|\sum_{n=1}^{\infty} x_n y_n\right| = \left|\lim_{k\to\infty} \sum_{n=1}^{k} x_n y_n\right|$$

$$= \lim_{k\to\infty} \left|\sum_{n=1}^{k} x_n y_n\right|$$

$$\le \lim_{k\to\infty} \sum_{n=1}^{k} |x_n y_n| = \sum_{n=1}^{\infty} |x_n y_n|$$

$$\le \|y\|_\infty \cdot \|x\|_1$$

So this Cauchy Schwarz like inequality between ℓ^1 and ℓ^∞ holds.

So we can conclude $\|\varphi_y\| \le \|y\|_\infty$