<u>Assignment 2 Question 4B</u>

(a) To show: $\mathrm{span}\{e_1, e_2, \ldots\} = c_c$.

We will use that c_c is a subspace of $\mathbb{R}^\infty$, which we might check separately.

To show: (aa) $\mathrm{span}\{e_1, e_2, \ldots\} \subseteq c_c$.

(ab) $c_c \subseteq \mathrm{span}\{e_1, e_2, \ldots\}$.

(aa) Since e_i has $x_n = 0$ for $n \in \mathbb{Z}_{>i}$ then $e_i \in c_c$.

If $x = c_1 e_1 + \cdots + c_k e_k$ is an element of $\mathrm{span}\{e_1, e_2, \ldots\}$ then

$x \in c_c$, since c_c is closed under finite sums and scalar multiplications (since c_c is a subspace).

So $\mathrm{span}\{e_1, e_2, \ldots\} \subseteq c_c$.

(ab) Let $x = (x_1, x_2, \ldots x_k, 0, 0, \ldots) \in c_c$ so that $x_n = 0$ for $n \in \mathbb{Z}_{>k}$.

Then

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_k e_k \in \mathrm{span}\{e_1, e_2, \ldots\}.$$

So $c_c \subseteq \mathrm{span}\{e_1, e_2, \ldots\}$.

So $c_c = \mathrm{span}\{e_1, e_2, \ldots\}$.

A. Ram

(b) Work in ℓ^p with the norm $\| \ \|_p$.

From part (a) $\mathrm{span}\{e_1, e_2, \ldots\} = c_c$.

To show: $\overline{c_c} = \ell^p$.

To show: (ba) $\overline{c_c} \subseteq \ell^p$

(bb) $\ell^p \subseteq \overline{c_c}$

(ba) By definition of closure in ℓ^p, $\overline{c_c} \subseteq \ell^p$.

(bb) Assume $x = (x_1, x_2, \ldots) \in \ell^p$.

To show: $x \in \overline{c_c}$.

To show: There exists a sequence $(w_1, w_2, \ldots)$ in c_c such that $\lim_{n\to\infty} w_n = x$.

Let

$$w_1 = (x_1, 0, 0, 0, \ldots)$$
$$w_2 = (x_1, x_2, 0, 0, \ldots)$$
$$w_3 = (x_1, x_2, x_3, 0, \ldots)$$
$$\vdots$$

To show: $\lim_{n\to\infty} w_n = x$.

~~To show: If $\varepsilon \in \mathbb{E}$ (where $\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}$) then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>N}$ then $\|w_n - x\|_p < \varepsilon$.~~

~~Assume $\varepsilon \in \mathbb{E}$.~~

To show: $\lim_{n\to\infty} \|x-w_n\|_p = 0$.

To show: $\lim_{n\to\infty} \|x-w_n\|_p^p = 0$.

Since $x - w_n = (0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots)$ then

$$\|x-w_n\|_p^p = \sum_{k=n+1}^{\infty} |x_k|^p = \sum_{k=1}^{\infty} |x_k|^p - \sum_{k=1}^{n} |x_k|^p$$

$$= \|x\|_p^p - \|w_n\|_p^p.$$

To show: $\lim_{n\to\infty} \left(\|x\|_p^p - \|w_n\|_p^p\right) = 0$.

To show: $\lim_{n\to\infty} \|w_n\|_p^p = \|x\|_p^p$.

This is true by definition of $\|x\|_p$, which says

$$\|x\|_p^p = \sum_{k=1}^{\infty} |x_k|^p = \lim_{n\to\infty} \left(\sum_{k=1}^{n} |x_k|^p\right) = \lim_{n\to\infty} \|w_n\|_p.$$

So $x = \lim_{n\to\infty} w_n$.

So $x \in \overline{c_c}$ and $\ell^p \subseteq \overline{c_c}$.

So $\overline{c_c} = \ell^p$.

(c) Work in ℓ^1 with the norm $\| \ \|_1$.

From part (a), $\mathrm{span}\{e_1, e_2, \ldots\} = c_c$.

To show: $\overline{c_c} = \ell^1$

To show: (ca) $\overline{c_c} \subseteq \ell^1$

(cb) $\ell^1 \subseteq \overline{c_c}$

(ca) By definition of closure in ℓ^1, $\overline{c_c} \subseteq \ell^1$.

(cb) Assume $x = (x_1, x_2, \ldots) \in \ell^1$.

To show: $x \in \overline{c_c}$.

To show: There exists a sequence $(w_1, w_2, \ldots)$ in c_c such that $\lim_{n\to\infty} w_n = x$.

Let
$$w_1 = (x_1, 0, 0, 0, \ldots)$$
$$w_2 = (x_1, x_2, 0, 0, \ldots)$$
$$w_3 = (x_1, x_2, x_3, 0, \ldots)$$
$$\vdots$$

To show: $\lim_{n\to\infty} w_n = x$.

To show: $\lim_{n\to\infty} \|x - w_n\|_1 = 0$.

Since $x - w_n = (0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots)$ then
$$\|x - w_n\|_1 = \sum_{k=n+1}^{\infty} |x_k| = \sum_{k=1}^{\infty} |x_k| - \sum_{k=1}^{n} |x_k|$$
$$= \|x\|_1 - \|w_n\|.$$

To show: $\lim_{n\to\infty} \|x\|_1 - \|w_n\|_1 = 0$.

To show: $\lim_{n\to\infty} \|w_n\|_1 = \|x\|_1$.

This is true by the definition of $\|x\|_1$, which says

$$\|x\|_1 = \sum_{k=1}^{\infty} |x_k| = \lim_{n\to\infty}\left(\sum_{k=1}^{n} |x_k|\right) = \lim_{n\to\infty} \|w_n\|_1.$$

So $x = \lim_{n\to\infty} w_n$.

So $x \in \overline{C_c}$. and $\ell^1 \subseteq \overline{C_c}$.

So $\overline{C_c} = \ell^1$.

A. Ram

(d) Work in ℓ^∞ with the norm $\| \;\|_\infty$.

To show: (da) $\overline{c_c} \subseteq c_0$

(db) $c_0 \subseteq \overline{c_c}$

(da) To show: If $x \in \overline{c_c}$ then $x \in c_0$.

Assume $x \in \overline{c_c}$ with $x = (x_1, x_2, \dots)$.

Then $x \in \ell^\infty$ and there exists a sequence $(w_1, w_2, \dots)$ in c_c such that $\lim_{n\to\infty} w_n = x$.

So $\lim_{n\to\infty} \|x - w_n\|_\infty = 0$.

To show: $x \in c_0$.

To show: $\lim_{n\to\infty} x_n = 0$.

To show: If $\varepsilon \in \mathbb{E}$ there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ then $|x_n| < \varepsilon$

Assume $\varepsilon \in \mathbb{E}$, where $\mathbb{E} = \{10^{-1}, 10^{-2}, \dots\}$.

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ then $|x_n| < \varepsilon$.

Since $\lim_{n\to\infty} \|x - w_n\|_\infty = 0$ then there exists $M_1 \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\geq M_1}$ then $\|x - w_k\|_\infty < \varepsilon$.

Since $w_{M_1} \in c_c$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then the n^{th} entry of w_M is 0.

So $\cancel{x - w_M}$ if $n \in \mathbb{Z}_{>N}$ then then n^{th} entry of $x - w_M$ is x_n. and

$$\varepsilon > \|x - w_M\|_\infty = \sup\{ |(x-w_M)_k| \mid k \in \mathbb{Z}_{>0}\}$$
$$\geq \sup\{ |(x-w_M)_k| \mid k \in \mathbb{Z}_{>N}\}$$
$$= \sup\{ |x_k| \mid k \in \mathbb{Z}_{>N}\}.$$

So, if $k \in \mathbb{Z}_{>N}$ then $|x_k| < \varepsilon$.

So $\lim_{n \to \infty} x_n = 0$ and $x \in c_0$.

So $\overline{c_c} \subseteq c_0$.

(bb) To show: $c_0 \subseteq \overline{c_c}$.

To show: If $x \in c_0$ then $x \in \overline{c_c}$.

Assume $x \in c_0$ with $x = (x_1, x_2, \ldots)$

To show: There exists a sequence $(w_1, w_2, \ldots)$ in c_c such that $\lim_{n \to \infty} w_n = x$.

Let
$$w_1 = (x_1, 0, 0, 0, \ldots)$$
$$w_2 = (x_1, x_2, 0, 0, \ldots)$$
$$w_3 = (x_1, x_2, x_3, 0, \ldots)$$
$$\vdots$$

To show: $\lim_{n \to \infty} w_n = x$.

To show: $\lim_{n \to \infty} \|x - w_n\|_\infty = 0$.

Since $x - w_n = (0, 0, \ldots 0, x_{n+1}, x_{n+2}, \ldots)$

then $\|x - w_n\|_\infty = \sup\{|x_k| \mid k \in \mathbb{Z}_{>n}\}$

We know that $\lim_{n\to\infty} x_n = 0$ since $x \in c_0$.

To show: $\lim_{n\to\infty} \|x - w_n\|_\infty = 0$.

To show: If $\varepsilon \in \mathbb{R}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>N}$ then $\|x - w_n\|_\infty < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}$.

Since $\lim_{n\to\infty} x_n = 0$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>N}$ then $|x_n| < \varepsilon$.

To show: If $n \in \mathbb{Z}_{>N}$ then $\|x - w_n\|_\infty < \varepsilon$.

Assume $n \in \mathbb{Z}_{>N}$.

To show: $\|x - w_n\|_\infty < \varepsilon$.

$$
\begin{aligned}
\|x - w_n\|_\infty &= \sup\{|x_k| \mid k \in \mathbb{Z}_{>n}\} \\
&\le \sup\{|x_k| \mid k \in \mathbb{Z}_{>N}\} \\
&\le \sup\{\varepsilon \mid k \in \mathbb{Z}_{>N}\} = \varepsilon.
\end{aligned}
$$

So $\lim_{n\to\infty} \|x - w_n\|_\infty = 0$ and $\lim_{n\to\infty} w_n = x$.

So $x \in \overline{c_c}$ and $c_0 \subseteq \overline{c_c}$

So $\overline{c_c} = c_0$.