Assignment 1 Question 3

This question is the same as Question 2 on Assignment 2 for Metric and Hilbert Spaces 2017 with solutions accessible from the web page for that course. Let us copy the solution from there, making a few small improvements.

(a) Let $\varphi: \mathbb{R}^2 \to \mathbb{R}$ be given by $\varphi(x_1, x_2) = ax_1 + bx_2$.

Assume $\|(x_1, x_2)\| = |x_1| + |x_2|$

To show: $\|\varphi\| = \max\{|a|, |b|\}$.

(aa) Let $(x_1, x_2) \in \mathbb{R}^2$. Then

$$\begin{aligned}|\varphi(x)| &= |ax_1 + bx_2| \leq |ax_1| + |bx_2| \\ &= |a||x_1| + |b||x_2| \\ &\leq \max\{|a|, |b|\} \cdot (|x_1| + |x_2|) \\ &= \max\{|a|, |b|\} \cdot \|x\|_1.\end{aligned}$$

So $\|\varphi\| = \sup\left\{ \dfrac{|\varphi(x)|}{\|x\|_1} \,\middle|\, x \in \mathbb{R}^2 \text{ and } x \neq 0 \right\} \leq \max\{|a|, |b|\}$

So $\|\varphi\| \leq \max\{|a|, |b|\}$.

(ab) Let $x = (1, 0)$. Then

$|\varphi(x)| = |ax_1 + 0x_2| = |ax_1| = |a| \, \|x\|_1$

So $\dfrac{|\varphi(x)|}{\|x\|_1} = |a|$.

So $\|\varphi\| = \sup\left\{ \dfrac{|\varphi(y)|}{\|y\|_1} \,\middle|\, y \in \mathbb{R}^2 \right\} \geq |a|$.

Let $x = (0,1)$. Then

$$|\varphi(x)| = |0x_1 + bx_2| = |b||x_2| = |b| \cdot \|x\|_1.$$

So $\dfrac{|\varphi(x)|}{\|x\|_1} = |b|$.

So $\|\varphi\| = \sup\left\{ \dfrac{|\varphi(y)|}{\|y\|_1} \,\middle|\, y \in \mathbb{R}^2 \right\} \geq |b|$.

So $\|\varphi\| \geq \max\{|a|, |b|\}$.

Combining (aa), which says $\|\varphi\| \leq \max\{|a|,|b|\}$ and (ab), which says $\|\varphi\| \geq \max\{|a|,|b|\}$ gives

$$\|\varphi\| = \max\{|a|,|b|\}.$$

(b) Let $\varphi: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$\varphi(x_1, x_2) = ax_1 + bx_2.$$

Assume $\|(x_1,x_2)\|_\infty = \max\{|x_1|, |x_2|\}$.

To show: $\|\varphi\| = |a| + |b|$.

To show: (ba) $\|\varphi\| \le |a| + |b|$

(bb) $\|\varphi\| \ge |a| + |b|$

(ba) Let $(x_1, x_2) \in \mathbb{R}^2$. Then

$$\begin{aligned} |\varphi(x)| &= |ax_1 + bx_2| \le |ax_1| + |bx_2| \\ &= |a||x_1| + |b||x_2| \\ &\le (|a|+|b|)\cdot \max\{|x_1|, |x_2|\} \\ &\le (|a|+|b|)\|x\|_\infty . \end{aligned}$$

So $$\frac{|\varphi(x)|}{\|x\|_\infty} \le |a| + |b|.$$

So $$\|\varphi\| = \sup\left\{ \frac{|\varphi(x)|}{\|x\|_\infty} \;\middle|\; \begin{matrix} x = (x_1,x_2) \in \mathbb{R}^2 \\ x \ne (0,0) \end{matrix} \right\} \le |a| + |b|.$$

(bb) To show: $\|\varphi\| \ge |a| + |b|$.

Let $$x = (x_1, x_2) = \begin{cases} (1,1), & \text{if } a \in \mathbb{R}_{\ge 0} \text{ and } b \in \mathbb{R}_{\ge 0} \\ (1,-1), & \text{if } a \in \mathbb{R}_{\ge 0} \text{ and } b \in \mathbb{R}_{< 0} \\ (-1,1), & \text{if } a \in \mathbb{R}_{< 0} \text{ and } b \in \mathbb{R}_{\ge 0} \\ (-1,-1), & \text{if } a \in \mathbb{R}_{< 0} \text{ and } b \in \mathbb{R}_{< 0} \end{cases}$$

Then $|x_1|=1$ and $|x_2|=1$ and

$$\begin{aligned}|\varphi(x)| &= ||a|+|b|| = |a|+|b| \\ &= (|a|+|b|)\cdot \max\{|x_1|,|x_2|\} \\ &= (|a|+|b|)\cdot \|x\|_\infty\end{aligned}$$

So $\|\varphi\| = \sup\left\{ \dfrac{|\varphi(y)|}{\|y\|_\infty} \,\middle|\, y\in\mathbb{R}^2, y\neq 0\right\} \geq |a|+|b|$.

Combining (ba), which says $\|\varphi\| \leq |a|+|b|$,
and (bb), which says $\|\varphi\| \geq |a|+|b|$,
gives $\|\varphi\| = |a|+|b|$.

(c) Assume $\|x\|_p = (|x_1|^p+|x_2|^p)^{1/p}$ and $\frac{1}{p}+\frac{1}{q}=1$.

Let $\varphi: \mathbb{R}^2 \to \mathbb{R}$ be given by $\varphi(x_1,x_2) = ax_1+bx_2$.

To show: $\|\varphi\| = (|x_1|^q+|x_2|^q)^{1/q}$

To show: (ca) $\|\varphi\| \leq (|x_1|^q+|x_2|^q)^{1/q}$

(cb) $\|\varphi\| \geq (|x_1|^q+|x_2|^q)^{1/q}$

(ca) Let $x=(x_1,x_2)$. Then

$$|\varphi(x)| = |ax_1+bx_2| \le (|a|^q+|b|^q)^{\frac{1}{q}}(|x_1|^p+|x_2|^p)^{1/p}$$

by the Hölder inequality proved as part (b) of Theorem 14.1 in Section 14.3 of the Function Spaces Notes on the course web page.

So $|\varphi(x)| \le (|a|^q+|b|^q)^{1/q}\|x\|_p$.

So $\dfrac{|\varphi(x)|}{\|x\|_p} \le (|a|^q+|b|^q)^{1/q}$

So $\|\varphi\| = \sup\left\{ \dfrac{|\varphi(x)|}{\|x\|_p} \,\Big|\, x\in\mathbb{R}^2, x\neq 0\right\} \le (|a|^q+|b|^q)^{1/q}$

(cb) To show: $\|\varphi\| \ge (|a|^q+|b|^q)^{1/q}$

Let $x = (\pm|a|^{q/p}, \pm|b|^{q/p})$. Then

$$|\varphi(x)| = (\pm 1)\,a\cdot|a|^{q/p} + (\pm 1)\,b\,|b|^{q/p}$$
$$= |a|^{1+q/p} + |b|^{1+q/p}$$
$$= |a|^{q(\frac{1}{q}+\frac{1}{p})} + |b|^{q(\frac{1}{q}+\frac{1}{p})}$$
$$= |a|^q+|b|^q$$
$$= (|a|^q+|b|^q)^{\frac{1}{q}+\frac{1}{p}}$$

(the signs ± 1 are chosen so that $(\pm 1)a\cdot|a|^{q/p} = |a|^{1+q/p}$ and $(\pm 1)b\cdot|b|^{q/p} = |b|^{1+q/p}$)

$$= (|a|^q + |b|^q)^{1/q} (|a|^q + |b|^q)^{1/p}$$

$$= (|a|^q + |b|^q)^{1/q} ((|a|^{q/p})^p + (|b|^{q/p})^p)^{1/p}$$

$$= (|a|^q + |b|^q)^{1/q} (|x_1|^p + |x_2|^p)^{1/p}$$

$$= (|a|^q + |b|^q)^{1/q} \|x\|_p.$$

So $\dfrac{|\varphi(x)|}{\|x\|_p} = (|a|^q + |b|^q)^{1/q}$.

So $\|\varphi\| = \sup\left\{ \dfrac{|\varphi(y)|}{\|y\|} \,\middle|\, y \in \mathbb{R}^2, y \neq 0 \right\} \geq (|a|^q + |b|^q)^{1/q}$

So (⁎a) gives $\|\varphi\| \leq (|a|^q + |b|^q)^{1/q}$ and
(⁎b) gives $\|\varphi\| \geq (|a|^q + |b|^q)^{1/q}$.

Thus $\|\varphi\| = (|a|^q + |b|^q)^{1/q}$.