MHS Assignment 1 Question 2

Let $\varepsilon \in \mathbb{R}_{>0}$. Assume that $\sum_{n=1}^{\infty} a_n \varepsilon^n$ exists in $\mathbb{C}$.

Assume $z \in B_\varepsilon(0)$ so that $z \in \mathbb{C}$ and $|z| < \varepsilon$.

To show: $\sum_{n=1}^{\infty} a_n z^n$ exists in $\mathbb{C}$.

Proposition 8.2 from the sequences and series notes on the course webpage says that

if $\sum_{n=1}^{\infty} |a_n z^n|$ exists in $\mathbb{C}$ then $\sum_{n=1}^{\infty} a_n z^n$ exists in $\mathbb{C}$.

To show: $\sum_{n=1}^{\infty} |a_n z^n|$ exists in $\mathbb{C}$.

Let $N \in \mathbb{Z}_{>0}$ be such that

if $n \in \mathbb{Z}_{\geq N}$ then $|a_n \varepsilon^n| < 1$.

(since $\sum_{n=1}^{\infty} a_n \varepsilon^n$ exists in $\mathbb{C}$ then $\lim_{n\to\infty} a \varepsilon^n = 0$).

Then

$$\sum_{n=1}^{\infty} |a_n z^n| = \sum_{n=1}^{N} |a_n z^n| + \sum_{n=N+1}^{\infty} |a_n z^n|$$

$$= \sum_{n=1}^{N} |a_n z^n| + \sum_{n=N+1}^{\infty} |a_n| \varepsilon^n \left|\frac{z}{\varepsilon}\right|^n$$

$$\leq \sum_{n=1}^{N} |a_n z^n| + \sum_{n=N+1}^{\infty} \left|\frac{z}{\varepsilon}\right|^n$$

$$= \sum_{n=1}^{N} |a_n z^n| + \left|\frac{z}{\epsilon}\right|^{N+1} \sum_{r=1}^{\infty} \left|\frac{z}{\epsilon}\right|^{r}$$

$$= \sum_{n=1}^{N} |a_n z^n| + \left|\frac{z}{\epsilon}\right|^{N+1} \frac{1}{1-\left|\frac{z}{\epsilon}\right|}.$$

So $\sum_{n=1}^{\infty} |a_n z^n|$ converges.

So, By Proposition 8.2, $\sum_{n=1}^{\infty} a_n z^n$ converges.