Assignment 1 Question 1

(a) Let $\theta \in \mathbb{R}$ with $0 \le \theta < 2\pi$.

An eigenvector $\begin{pmatrix}1\\b\end{pmatrix}$ of eigenvalue λ has

$b \in \mathbb{C}$ and $\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1\\b\end{pmatrix} = \lambda\begin{pmatrix}1\\b\end{pmatrix}$.

So $\begin{pmatrix}\cos\theta-\lambda & \sin\theta\\ -\sin\theta & \cos\theta-\lambda\end{pmatrix}\begin{pmatrix}1\\b\end{pmatrix} = 0$

Since $\begin{pmatrix}1\\b\end{pmatrix} \in \ker\begin{pmatrix}\cos\theta-\lambda & \sin\theta\\ -\sin\theta & \cos\theta-\lambda\end{pmatrix}$ then

$\ker\begin{pmatrix}\cos\theta-\lambda & \sin\theta\\ -\sin\theta & \cos\theta-\lambda\end{pmatrix} \neq 0$ and $\begin{pmatrix}\cos\theta-\lambda & \sin\theta\\ -\sin\theta & \cos\theta-\lambda\end{pmatrix}$

is not invertible (since it is not injective).

So

$$\begin{aligned}
0 &= \det\begin{pmatrix}\cos\theta-\lambda & \sin\theta\\ -\sin\theta & \cos\theta-\lambda\end{pmatrix}\\
&= \cos^2\theta - 2\cos\theta\,\lambda + \lambda^2 + \sin^2\theta\\
&= \lambda^2 - 2\cos\theta\,\lambda + 1 = (\lambda-\cos\theta)^2 + 1 - \cos^2\theta\\
&= (\lambda-\cos\theta)^2 + \sin^2\theta.
\end{aligned}$$

So $(\lambda-\cos\theta)^2 = -\sin^2\theta$ and $\lambda - \cos\theta = \pm i\sin\theta$

The equation

$\begin{pmatrix}\cos\theta-\lambda & \sin\theta\\ -\sin\theta & \cos\theta-\lambda\end{pmatrix}\begin{pmatrix}1\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ gives $(\cos\theta-\lambda) + b\sin\theta = 0$ and $-\sin\theta + b(\cos\theta-\lambda) = 0$.

So,

if $\lambda = \cos\theta + i\sin\theta$ then $b = \frac{\sin\theta}{-i\sin\theta} = i$, and

if $\lambda = \cos\theta - i\sin\theta$ then $b = \frac{\sin\theta}{i\sin\theta} = -i$.

So $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector with eigenvalue

$$\lambda = \cos\theta + i\sin\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} + i\,\frac{e^{i\theta}-e^{-i\theta}}{2i}$$
$$= e^{i\theta}$$

and

$\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector with eigenvalue

$$\lambda = \cos\theta - i\sin\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} - i\,\frac{e^{i\theta}-e^{-i\theta}}{2i}$$
$$= e^{-i\theta}$$

(b) By part (a), the eigenvectors of

$\begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}$ are $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -i \end{pmatrix}$,

neither of which are in $\mathbb{R}^2$

The eigenvalues are

$$e^{i\pi/4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \quad \text{and} \quad e^{-i\pi/4} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2},$$

neither of which are in $\mathbb{R}$.

So $\begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ does not have an eigenvector in $\mathbb{R}^2$.

(c) Let $n \in \mathbb{Z}_{>0}$ and let $A \in M_n(\mathbb{C})$ A. Ram

To show: There exists $v \in \mathbb{C}^n$ which is an eigenvector of A.

To show: There exists $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$ with

$$v \neq 0 \text{ and } Av = \lambda v.$$

To show: There exists $\lambda \in \mathbb{C}$ such that

$$\ker(A - \lambda) \neq 0.$$

Let $p(t) = \det(A - t) \in \mathbb{C}[t]$.

Since $\mathbb{C}$ is algebraically closed then there exist $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ such that

$$p(t) = (\lambda_1 - t)(\lambda_2 - t)\cdots(\lambda_n - t).$$

Let $\lambda = \lambda_1$

Then $\det(A - \lambda) = (\lambda_1 - \lambda_1)(\lambda_2 - \lambda_1)\cdots(\lambda_n - \lambda_1)$

$$= 0 \cdot (\lambda_2 - \lambda_1)\cdots(\lambda_n - \lambda_1) = 0$$

So $\ker(A - \lambda) \neq 0$.

So there exists $v \in \ker(A - \lambda)$ with $v \neq 0$.

So there exists $v \in \mathbb{C}^n$ with $v \neq 0$ and

$$(A - \lambda)v = 0$$

So there exists $v \in \mathbb{C}^n$ with $v \neq 0$ and

$$Av = \lambda v.$$

So A has an eigenvector in $\mathbb{C}^n$.