3 Lecture 3, 9 March 2022: The double affine Hecke algebra (DAHA)

3.1 Page 1: Presentation of the DAHA

The *double affine Hecke algebra (of type GL_n)* is the algebra generated by symbols g and x_k and T_i for $i, k \in \mathbb{Z}$ with relations

$$T_{i+n} = T_i, \qquad x_{i+n} = q^{-1}x_i, \qquad x_k x_\ell = x_\ell x_k, \qquad \text{for } i, k, \ell \in \mathbb{Z}; \qquad \text{(periodicityrelsF)}$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad T_iT_j = T_jT_i, \qquad T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})T_i + 1, \qquad \text{(HeckerelsF)}$$

for $i, j \in \mathbb{Z}$ with $j \notin \{i-1, i+1\}$;

$$\begin{matrix} T_ix_i = x_{i+1}T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})x_{i+1}, \\ T_ix_{i+1} = x_iT_i + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})x_{i+1}, \end{matrix} \qquad x_{i+1} = T_ix_iT_i, \qquad \text{and} \qquad T_ix_j = x_jT_i, \qquad \text{(XaffHeckerelsF)}$$

for $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, n\}$ with $j \notin \{i, i+1\}$; and

$$gx_i = x_{i+1}g \qquad \text{and} \qquad gT_i = T_{i+1}g \qquad \text{for } i \in \mathbb{Z}. \qquad \text{(DAHArels2F)}$$

Proposition 3.1. *(The glue relations) Define*

$$g^\vee = x_1T_1 \cdots T_{n-1}.$$

Then

$$T_1^{-1}gg^\vee = g^\vee gT_{n-1} \qquad \textit{and} \qquad T_{n-1}^{-1} \cdots T_1^{-1}g(g^\vee)^{-1} = q(g^\vee)^{-1}gT_{n-1} \cdots T_1.$$

3.2 Page 2: Cherednik-Dunkl operators

The *Cherednik-Dunkl operators* are $Y_1, \ldots, Y_n$ given by

$$Y_1 = gT_{n-1} \cdots T_1, \qquad \text{and} \qquad Y_{j+1} = T_j^{-1}Y_jT_j^{-1} \quad \text{for } j \in \{1, \ldots, n-1\}. \qquad \text{(CDops)}$$

These are analogues of Murphy elements in the DAHA. The following proposition shows that these form a family of commuting operators.

Proposition 3.2. *If $i, j \in \{1, \ldots, n\}$ then $Y_iY_j = Y_jY_i$.*

3.3 Page 3: Intertwiners

The intertwiners $\tau_1^\vee, \ldots, \tau_{n-1}^\vee$ are defined by

$$\tau_i^\vee = T_i + \frac{(t^{-\frac{1}{2}} - t^{\frac{1}{2}})}{1 - Y_i^{-1}Y_{i+1}} = T_i^{-1} + \frac{(t^{-\frac{1}{2}} - t^{\frac{1}{2}})Y_i^{-1}Y_{i+1}^{-1}}{1 - Y_i^{-1}Y_{i+1}}, \qquad \text{(tauiops)}$$

where the second equality follows from $T_i^{-1} = T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$. The intertwiner $\tau_\pi^\vee$ is defined by

$$\tau_\pi^\vee = x_1T_1 \cdots T_{n-1}. \qquad \text{(taupiop)}$$

The following Proposition determines how the intertwiners $\tau_i^\vee$ and $\tau_g^\vee$ move past the Y_j.

Proposition 3.3. *If $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, n\}$ then*

$$\tau_i^\vee Y_i = Y_{i+1}\tau_i^\vee, \qquad \tau_i^\vee Y_{i+1} = Y_i\tau_i^\vee, \qquad \textit{and} \qquad \tau_i^\vee Y_j = Y_j\tau_i^\vee \quad \textit{if } j \notin \{i, i+1\}. \qquad \text{(taupastYrels1)}$$

If $j \in \{1, \ldots, n-1\}$ then

$$\tau_\pi^\vee Y_j = Y_{j+1}\tau_\pi^\vee \qquad \textit{and} \qquad \tau_\pi^\vee Y_n = q^{-1}Y_1\tau_\pi^\vee. \qquad \text{(taupastYrels2)}$$

3.4 Page 4: DAHA acts on polynomials

This subsection defines **the polynomial representation** of the DAHA.

Let $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The symmetric group S_n acts on $\mathbb{C}[X]$ by permuting $x_1, \ldots, x_n$. Letting $s_1, \ldots, s_{n-1}$ denote the *simple transpositions* in S_n,

$$(s_i f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n). \qquad \text{(siops)}$$

For $j \in \{1, \ldots, n\}$ define operators $y_1, \ldots, y_n$ by

$$(y_i f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, q^{-1}x_i, x_{i+1}, \cdots x_n). \qquad \text{(yjops)}$$

For $f \in \mathbb{C}[X]$ and $i \in \{1, \ldots, n-1\}$ define the *divided difference operators* $\partial_i \colon \mathbb{C}[X] \to \mathbb{C}[X]$ and the *Hecke algebra operators* $T_i \colon \mathbb{C}[X] \to \mathbb{C}[X]$ and the *promotion operator* $g \colon \mathbb{C}[X] \to \mathbb{C}[X]$ by

$$\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}, \qquad T_i = t^{-\frac{1}{2}} x_{i+1}\partial_i - t^{\frac{1}{2}}\partial_i x_{i+1} \qquad \text{and} \qquad g = s_1 \cdots s_{n-1} y_n, \qquad \text{(divdiffops)}$$

For $i \in \{1, \ldots, n\}$ let $X_i \colon \mathbb{C}[X] \to \mathbb{C}[X]$ be the operator given by multiplication by x_i (i.e. $X_i f = x_i f$ for $f \in \mathbb{C}[X]$).

Theorem 3.4. *The formulas (divdiffops) define an action of the double affine Hecke algebra on* $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

A way of deriving the formulas in (divdiffops) is to consider the induced representation

$$\mathbb{C}[X] = \mathrm{Ind}_{H_Y}^{\widetilde{H}}(\mathbf{1}_Y) = \mathbb{C}\text{-span}\{x_1^{\mu_1} \cdots x_n^{\mu_n} \mathbf{1}_Y \mid \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n\}$$

determined by

$$g\mathbf{1}_Y = \mathbf{1}_Y \qquad \text{and} \qquad T_i \mathbf{1}_Y = t^{\frac{1}{2}} \mathbf{1}_Y.$$

Then the formulas in (divdiffops) are consequences of the relations in (XaffHeckerelsF) and (DAHArels2F).

Remark 3.5. An alternate expression for ∂_i is

$$\partial_i = (1 + s_i)\frac{1}{x_i - x_{i+1}},$$

which is the form in which ∂_i arises as a push-pull operator in cohomology of the flag variety. The Leibniz rule for ∂_i is

$$\partial_i(f_1 f_2) = (\partial_i f_1) f_2 + (s_i f_1)(\partial_i f_2),$$

and the 0-Hecke algebra relations are

$$\partial_i^2 = 0, \qquad \partial_i \partial_j = \partial_j \partial_i, \qquad \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1},$$

for $i, j \in \{1, \ldots, n-1\}$ and $j \notin \{i+1, i-1\}$. All of these identities for the operators ∂_i are verified by direct computation. In particular,

$$\partial_1 \partial_2 \partial_1 = \frac{1}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)} \sum_{w \in S_3} w,$$

which is a formula for the push forward $H_T^*(Fl_3) \to H_T^*(\mathrm{pt})$ where Fl_3 denotes the full flag variety in $\mathbb{C}^3$. □

3.5 Page 5: c-functions

3.5.1 c-functions in xs

Let

$$c_{ij}(x) = \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} x_i x_j^{-1}}{1 - x_i x_j^{-1}} = t^{-\frac{1}{2}} \frac{x_j - t x_i}{x_j - x_i}, \qquad \text{for } i, j \in \{1, \ldots, n\} \text{ with } i \neq j. \qquad \text{(cfnxdefn)}$$

As operators on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$,

$$T_i = c_{i,i+1}(x) s_i - (c_{i,i+1}(x) - t^{\frac{1}{2}}), \qquad \text{(Tiviacfcn)}$$

Another formula for the action of T_i is

$$t^{\frac{1}{2}} T_i = t - \frac{t x_i - x_{i+1}}{x_i - x_{i+1}} (1 - s_i), \qquad \text{(TiviaBLop)}$$

3.5.2 c-functions in Ys

Let

$$c_{ij}(Y) = \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} Y_i Y_j^{-1}}{1 - Y_i Y_j^{-1}} = t^{-\frac{1}{2}} \frac{Y_j - t Y_i}{Y_j - Y_i}, \qquad \text{for } i, j \in \{1, \ldots, n\} \text{ with } i \neq j. \qquad \text{(cfnYdefn)}$$

Letting

$$\eta_{s_i} = \tau_i^{\vee} \frac{1}{c_{i,i+1}(Y)} \qquad \text{then} \qquad T_i = \eta_{s_i} c_{i,i+1}(Y) - (c_{i+1,i}(Y) - t^{\frac{1}{2}}), \qquad \text{(TiviacfcnY)}$$

The striking similarity between (Tiviacfcn) and (TiviacfcnY) is the core of the XY-parallelism in double affine Artin groups and double affine Hecke algebras (see [Mac03, §3.5]).