18.05.2022
Lecture 12 ①
A. Ram

The DAWG $\widetilde{W}$

Define $\langle , \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ and $|\cdot|^2 : \mathbb{Z}^n \to \mathbb{Z}$ by

$$\langle \lambda, \mu \rangle = \lambda_1\mu_1 + \cdots + \lambda_n\mu_n \quad \text{and} \quad |\lambda|^2 = \langle \lambda, \lambda \rangle,$$

for $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ in $\mathbb{Z}^n$.

The symmetric group S_n acts on $\mathbb{Z}^n$ by permuting the coordinates.

The <u>double</u> <u>affine</u> <u>Weyl group</u> $\widetilde{W}$ is generated by

$$q, \quad w \text{ for } w \in S_n, \quad x_\mu \text{ for } \mu \in \mathbb{Z}^n, \quad y_\lambda \text{ for } \lambda \in \mathbb{Z}^n,$$

with relations

(a) S_n is a subgroup of $\widetilde{W}$,

(b) $x_\mu x_\nu = x_{\mu+\nu}$ and $y_\lambda y_\gamma = y_{\lambda+\gamma}$,

(c) $w x_\mu = x_{w\mu} w$ and $w y_\lambda = y_{w\lambda} w$

(d) $q \in Z(\widetilde{W})$ and $x_\mu y_\lambda = q^{\langle \mu, \lambda \rangle} y_\lambda x_\mu$

The <u>affine Weyl group W_X</u> is the subgroup of $\widetilde{W}$ generated by w for $w \in S_n$ and x_μ for $\mu \in \mathbb{Z}^n$.

The <u>affine Weyl group W_Y</u> is the subgroup of $\widetilde{W}$ generated by w for $w \in S_n$ and y_λ for $\lambda \in \mathbb{Z}^n$

The <u>Heisenberg group D</u> is the subgroup of $\widetilde{W}$ generated by q, x_μ for $\mu \in \mathbb{Z}^n$, y_λ for $\lambda \in \mathbb{Z}^n$.

Matrix representation of $\tilde{W}$

$$x_\mu = \left(\begin{array}{c|c|c} \begin{matrix}1&0\\0&1\end{matrix} & \begin{matrix}-\mu^t\\0\end{matrix} & \begin{matrix}-\frac{1}{2}|\mu|^2 & 0\\ 0 & 0\end{matrix} \\ \hline 0 & 1 & \begin{matrix}\mu & 0\end{matrix} \\ \hline 0 & 0 & \begin{matrix}1&0\\0&1\end{matrix}\end{array}\right) \qquad y_\lambda = \left(\begin{array}{c|c|c} \begin{matrix}1&0\\0&1\end{matrix} & \begin{matrix}0\\ \lambda^t\end{matrix} & \begin{matrix}0 & 0\\ 0 & -\frac{1}{2}|\lambda|^2\end{matrix} \\ \hline 0 & 1 & \begin{matrix}0 & -\lambda\end{matrix} \\ \hline 0 & 0 & \begin{matrix}1&0\\0&1\end{matrix}\end{array}\right)$$

$$q^k = \left(\begin{array}{c|c|c} \begin{matrix}1&0\\0&1\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}0 & k\\ -k & 0\end{matrix} \\ \hline 0 & 1 & \begin{matrix}0 & 0\end{matrix} \\ \hline 0 & 0 & \begin{matrix}1&0\\0&1\end{matrix}\end{array}\right) \qquad w = \left(\begin{array}{c|c|c} \begin{matrix}1&0\\0&1\end{matrix} & \begin{matrix}0\\0\end{matrix} & 0 \\ \hline 0 & w & 0 \\ \hline 0 & 0 & \begin{matrix}1&0\\0&1\end{matrix}\end{array}\right)$$

The affine Weyl group W

Define

$$t_\lambda = q^{-\frac{1}{2}|\lambda|^2} x_\lambda y_\lambda \qquad \text{for } \lambda \in \mathbb{Z}^n.$$

Then

$$t_\lambda t_\mu = t_{\lambda+\mu} \quad \text{and} \quad w t_\lambda = t_{w\lambda} w.$$

The affine Weyl group W is the subgroup of $\tilde{W}$ generated by

$$t_\lambda \text{ for } \lambda \in \mathbb{Z}^n \quad \text{and} \quad w \text{ for } w \in S_n.$$

Chamber generators

For $j \in \{1, \ldots, n\}$ let

$$\varepsilon_j = (0, \ldots 0, 1, 0, \ldots 0) \text{ with } 1 \text{ in the } j^{th} \text{ entry}$$

For $i \in \{1, \ldots, n-1\}$ let

$s_i \in S_n$ be the transposition switching i and $i+1$.

Define

$$\theta = \varepsilon_1 - \varepsilon_n \quad \text{and} \quad s_\theta = s_1 \cdots s_{n-1} \cdots s_1.$$

Define

$$s_\pi = y_{\varepsilon_1} s_1 \cdots s_{n-1}, \qquad s_\pi^\sharp = t_{\varepsilon_1} s_1 \cdots s_{n-1}, \qquad s_\pi^\vee = x_{\varepsilon_1} s_1 \cdots s_{n-1}$$

$$s_0 = y_\theta s_\theta, \qquad s_0^\sharp = t_\theta s_\theta, \qquad s_0^\vee = x_\theta s_\theta$$

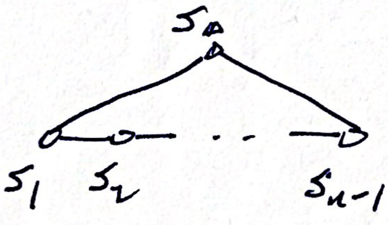
Set

$$s_i = s_i^\sharp = s_i^\vee \quad \text{for } i \in \{1, \ldots, n-1\}$$

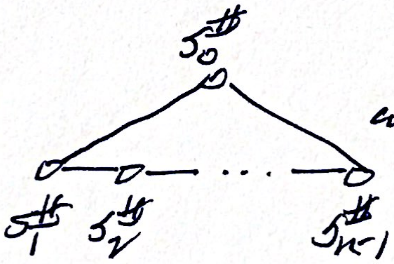
and define

$$s_i = s_{i+n}, \qquad s_i^\sharp = s_{i+n}^\sharp, \qquad s_i^\vee = s_{i+n}^\vee$$

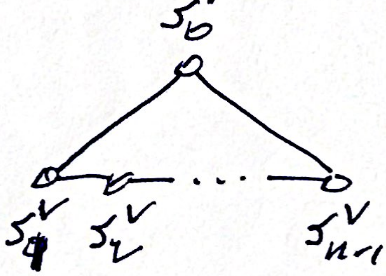
for $i \in \mathbb{Z}$. Then



and



and



where $\overset{a}{\circ}\!-\!\overset{b}{\circ}$ means $aba = bab$

and $\overset{a}{\circ} \ \overset{b}{\circ}$ means $ab = ba$.

$$s_i^2 = 1 \quad \text{and} \quad (s_i^\sharp)^2 = 1 \quad \text{and} \quad (s_i^\vee)^2 = 1$$

18.05.2022
Lecture 12 3.5

$$s_\pi s_i (s_\pi)^{-1} = s_{i+1}, \quad s_\pi^\# s_i^\# (s_\pi^\#)^{-1} = s_{i+1}^\#, \quad s_\pi^\vee s_i^\vee (s_\pi^\vee)^{-1} = s_{i+1}^\vee$$

and the glue relations are

$$s_1 s_\pi^\vee s_\pi = s_\pi s_\pi^\vee s_{n-1}$$

$$s_{n-1} \cdots s_1 s_\pi (s_\pi^\vee)^{-1} = q (s_\pi^\vee)^{-1} s_\pi s_{n-1} \cdots s_1$$

and the superglue relations are

$$(s_\pi^\vee)^{-1} s_\pi^\# (s_\pi)^{-1} s_1 \cdots s_{n-1} = q^{-\frac{1}{2}}$$

$$(s_0^\vee)^{-1} s_0^\# (s_0)^{-1} s_\theta = q^{-1}.$$

The DAArt $\widehat{B}$

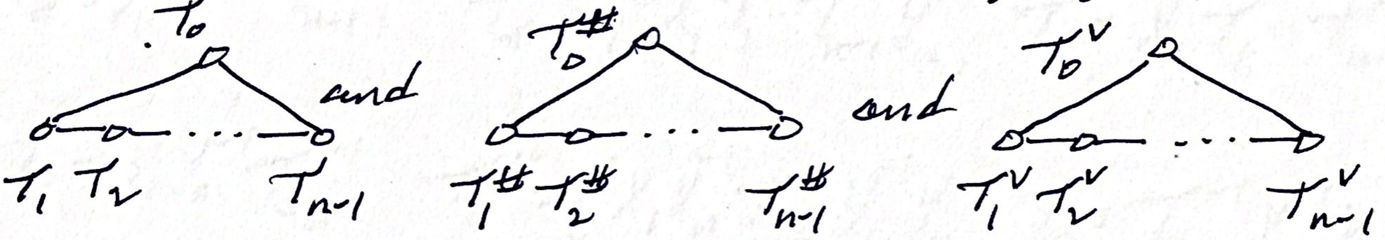
The group $\widehat{B}$ is presented by

q, $T_\pi, T_\pi^\#, T_\pi^\vee$ and $T_i, T_i^\#, T_i^\vee$ for $i \in \mathbb{Z}$

with relations

$q \in Z(\widehat{B})$, $\quad T_i = T_{i+n}$, $T_i^\# = T_{i+n}^\#$, $T_i^\vee = T_{i+n}^\vee$ for $i \in \mathbb{Z}$

$$T_i = T_i^\# = T_i^\vee \quad \text{for } i \in \{1, \dots, n-1\}$$



and with the glue relations

$$T_1 T_\pi^\vee T_\pi = T_\pi T_\pi^\vee T_{n-1}^{-1}$$

$$T_{n-1}^{-1} \cdots T_1^{-1} T_\pi (T_\pi^\vee)^{-1} = q (T_\pi^\vee)^{-1} T_\pi T_{n-1} \cdots T_1$$

and the superglue relations

$$(T_\pi^\vee)^{-1} T_\pi^\# (T_\pi)^{-1} T_1 \cdots T_{n-1} = q^{-\frac{1}{2}}$$

$$(T_0^\vee)^{-1} T_0^\# (T_0)^{-1} T_1 \cdots T_{n-1} \cdots T_1 = q^{-1}.$$

Define $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_n$ by

$$Y_1 = T_\pi T_1\cdots T_{n-1} \quad\text{and}\quad Y_j = T_{j-1}^{-1}Y_{j-1}T_{j-1}^{-1}$$

$$X_1 = T_\pi^\vee T_1^{-1}\cdots T_{n-1}^{-1} \quad\text{and}\quad X_j = T_{j-1}X_{j-1}T_{j-1}$$

then

$$X_iX_j = X_jX_i \quad\text{and}\quad Y_iY_j = Y_jY_i \quad\text{for } i,j\in\{1,\ldots,n\},$$

$$T_\pi = Y_1T_1^{-1}\cdots T_{n-1}^{-1},\quad T_\pi^\# = q^{-\frac12}X_1T_1\cdots T_{n-1}Y_n,\quad T_\pi^\vee = X_1T_1\cdots T_{n-1}$$

$$T_0 = Y_1Y_n^{-1}T_{s_\theta}^{-1},\quad T_0^\# = q^{-1}X_1X_n^{-1}T_{s_\theta}Y_1^{-1}Y_n,\quad T_0^\vee = T_{s_\theta}^{-1}X_1^{-1}X_n$$

where $T_{s_\theta} = T_1\cdots T_{n-1}\cdots T_1$.

The group $GL_2(\mathbb{Z})$

$$GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \;\middle|\; \begin{matrix} a,b,c,d \in \mathbb{Z} \\ ad-bc \in \mathbb{Z}^\times \end{matrix} \right\}$$

where $\mathbb{Z}^\times = \{1,-1\}$ and the operation is matrix multiplication. Let

$$\sigma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proposition

(a) The group $GL_2(\mathbb{Z})$ is presented by generators σ_1, σ_2 and s with relations

$$(\sigma_1\sigma_2\sigma_1)^2 \in Z(GL_2(\mathbb{Z})), \qquad \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

$$s^2 = 1, \qquad s\sigma_1 = \sigma_2^{-1}s, \qquad s\sigma_2 = \sigma_1^{-1}s$$

(b) The group $GL_2(\mathbb{Z})$ is presented by generators γ_1, γ_2 and s with relations

$$\gamma_1^4 = 1, \qquad \gamma_2^3 = \gamma_1^2, \qquad \gamma_1^2 = Z(GL_2(\mathbb{Z}))$$

$$s^2 = 1, \qquad s\gamma_1 = \gamma_1^{-1}s, \qquad s\gamma_2 = \gamma_2^{-1}s.$$

$GL_2(\mathbb{Z})$ acts by automorphisms

Define an injective homomorphism

$$GL_2(\mathbb{Z}) \longrightarrow GL_{n+4}(\mathbb{Z})$$

$$u \longmapsto \left(\begin{array}{c|c|c} u & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & (u^{-1})^t \end{array}\right)$$

If $u = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL_2(\mathbb{Z})$ then

$$u x_\mu u^{-1} = q^{\frac{1}{2}ad|\mu|^2} x_{a\mu} y_{-b\mu}, \qquad u W u^{-1} = W,$$

$$u y_\lambda u^{-1} = q^{\frac{1}{2}cd|\lambda|^2} x_{-c\lambda} y_{d\lambda}, \qquad u q u^{-1} = q$$

and

$$\sigma_1 s_0 \sigma_1^{-1} = (s_0)^{-1} s_0^\# s_0, \qquad \sigma_2 s_0 \sigma_2^{-1} = s_0, \qquad s s_0 s^{-1} = W_0 s_0^\vee W_0^{-1}$$

$$\sigma_1 s_0^\# \sigma_1^{-1} = s_0, \qquad \sigma_2 s_0^\# \sigma_2^{-1} = (s_0^\#)^{-1} s_0^\vee s_0^\#, \qquad s s_0^\# s^{-1} = W_0 s_0^\# W_0^{-1}$$

$$\sigma_1 s_0^\vee \sigma_1^{-1} = s_0^\vee, \qquad \sigma_2 s_0^\vee \sigma_2^{-1} = s_0^\#, \qquad s s_0^\vee s^{-1} = W_0 s_0 W_0^{-1}$$

and

$$\sigma_1 s_\pi \sigma_1^{-1} = q^{\frac{1}{2}} x_{-\varepsilon_1} s_\pi \qquad \sigma_2 s_\pi \sigma_2^{-1} = s_\pi, \qquad s s_\pi s^{-1} = W_0 (s_\pi^\vee)^{-1} W_0^{-1}$$

$$\sigma_1 s_\pi^\# \sigma_1^{-1} = s_\pi \qquad \sigma_2 s_\pi^\# \sigma_2^{-1} = y_{\varepsilon_1} s_\pi^\# \qquad s s_\pi^\# s^{-1} = W_0 (s_\pi^\#)^{-1} W_0^{-1}$$

$$\sigma_1 s_\pi^\vee \sigma_1^{-1} = s_\pi^\vee \qquad \sigma_2 s_\pi^\vee \sigma_2^{-1} = s_\pi^\#, \qquad s s_\pi^\vee s^{-1} = W_0 (s_\pi)^{-1} W_0^{-1}$$

18.05.2022
Lecture 12 (5.5)
A. Ram

$SL_2(\mathbb{Z})$ is the subgroup of $GL_2(\mathbb{Z})$ generated by σ_1 and σ_2

$SL_2(\mathbb{Z})$ is the subgroup of $GL_2(\mathbb{Z})$ generated by γ_1 and γ_2.

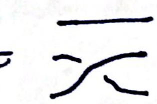
The <u>braid group on 3 strands</u> B_3 is presented by generators b_1 and b_2 with relations

$$b_1 b_2 b_1 = b_2 b_1 b_2 .$$

Then

$Z(B_3)$ is the subgroup generated by $(\sigma_1\sigma_2\sigma_1)^2$.

The pictures

$\sigma_1 =$  and $\sigma_2 =$ 

are a realization of B_3 as a fundamental group

$$B_3 \cong \pi_1\left(\frac{\mathbb{C}^3 - (H_{12} \cup H_{13} \cup H_{23})}{S_3}\right) \quad \text{where}$$

$H_{12} = \{(a_1, a_1, a_3) \in \mathbb{C}^3\}$

$H_{13} = \{(a_1, a_2, a_1) \in \mathbb{C}^3\}$

$H_{23} = \{(a_1, a_2, a_2) \in \mathbb{C}^3\}$

and the symmetric group S_3 acts by permuting the coordinates of $\mathbb{C}^3$.

18.05.2022
Lecture 12 5.75
A. Ram

The group $SL_2(\mathbb{Z})$ acts on the upper half plane

$$G_1 = \{ z \in \mathbb{C} \mid Im(z) \in \mathbb{R}_{>0} \} \text{ by } \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot z = \frac{az+c}{bz+d}$$

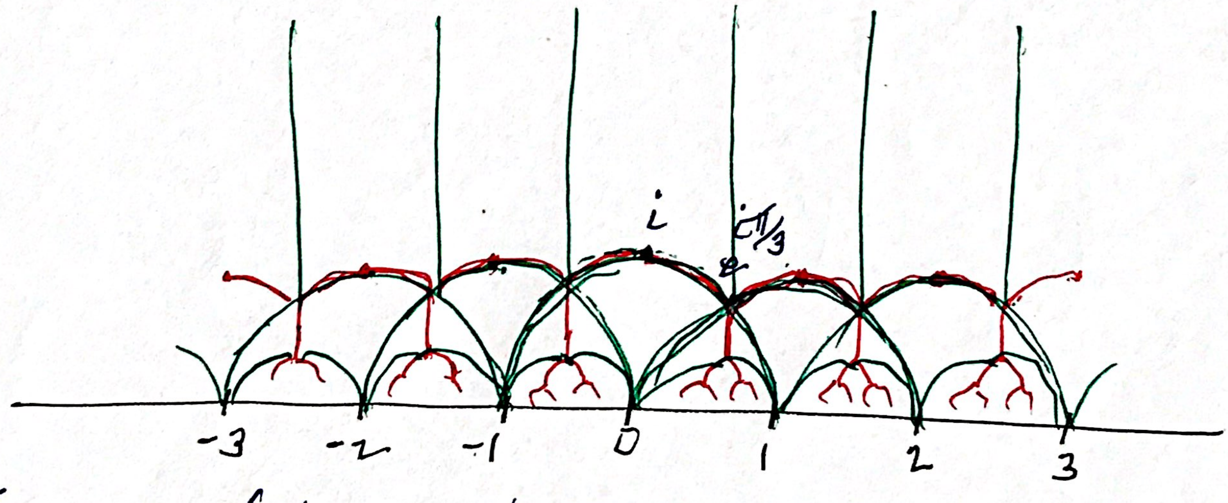
The stabilizer of the point i is

$$Stab(i) = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle = \langle \gamma_1 \rangle \cong \mathbb{Z}/4\mathbb{Z}.$$

The stabilizer of the point $e^{i\pi/3}$ is

$$Stab(e^{i\pi/3}) = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle = \langle \gamma_2 \rangle \cong \mathbb{Z}/6\mathbb{Z}$$

An infinite three valent tree is generated by the action of $SL_2(\mathbb{Z})$ on the arc of the unit circle connecting i and $e^{i\pi/3}$.



The arc of the unit circle connecting i and $e^{i\pi/3}$ (in fact every element of G_1) is stabilized by $-1 \in SL_2(\mathbb{Z})$