04.05.2022
Lecture 10 (1)
A. Ram

Product formulas for Macdonald polynomials

Electronic structure constants

The structure constants for the algebra $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with respect to the basis

$\{E_\mu \mid \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n\}$ of electronic Macdonald polynomials

are $a_{\mu\nu}^{\gamma}$ given by

$$E_\mu E_\nu = \sum_\gamma a_{\mu\nu}^{\gamma} E_\gamma .$$

Generating products

Let $\varepsilon_j = (0, \ldots, 0, 1, 0, \ldots 0)$ with 1 in the j^{th} entry.

$x_1, \ldots, x_n$ are generators of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Find $A_{\mu j}^{\gamma}, B_{\mu j}^{\gamma}, C_{\mu j}^{\gamma}, D_{\mu j}^{\gamma}, F_{\mu j}^{\gamma}, G_{\mu j}^{\gamma}$ given by

$$x_j E_\mu = \sum_\gamma A_{\mu j}^{\gamma} E_\gamma \qquad x_j^{-1} E_\mu = \sum_\gamma B_{\mu j}^{\gamma} E_\gamma$$

$$(x_1 + \cdots + x_j) E_\mu = \sum_\gamma C_{\mu j}^{\gamma} E_\gamma \qquad (x_1^{-1} + \cdots + x_j^{-1}) E_\mu = \sum_\gamma D_{\mu j}^{\gamma} E_\gamma$$

$$E_{\varepsilon_j} E_\mu = \sum_\gamma F_{\mu j}^{\gamma} E_\gamma \qquad E_{-\varepsilon_j} E_\mu = \sum_\gamma F_{\mu j}^{\gamma} E_\gamma$$

In analogy with Schubert calculus, call these products <u>Monk rules</u>.

Bosonic structure constants

The symmetric group S_n acts on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by permuting $x_1, \ldots, x_n$.

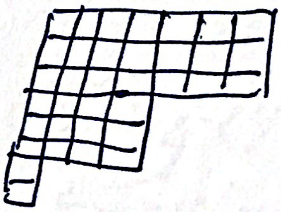
$$\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n} = \left\{ f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \;\middle|\; \begin{matrix}\text{if } w \in S_n \text{ then} \\ wf = f\end{matrix} \right\}$$

The structure constants for the algebra $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n}$ with respect to the basis

$$\left\{ P_\lambda \;\middle|\; \begin{matrix}\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \\ \lambda_1 \geq \cdots \geq \lambda_n\end{matrix} \right\}$$ of bosonic Macdonald polynomials

are $c_{\mu\nu}^\lambda$ given by

$$P_\mu P_\nu = \sum_\lambda c_{\mu\nu}^\lambda P_\lambda$$

<u>Generating products</u> Draw $\lambda =$ 
with λ_i boxes in row i.

$P_{\square}, P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}, P_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}, \ldots$ generate $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n}$

$P_{\square}, P_{\square\square}, P_{\square\square\square}, \ldots$ generate $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n}$

Find $K_{\mu j}^\lambda$ and $L_{\mu j}^\lambda$ given by

$$P_{r\left\{\begin{smallmatrix}\square\\\vdots\\\square\end{smallmatrix}\right.} P_\mu = \sum_\lambda K_{\mu j}^\lambda P_\lambda \quad \text{and} \quad P_{\underbrace{\square\cdots\square}_{r}} P_\mu = \sum_r L_{\mu r}^\lambda P_\lambda$$

Pieri formulas

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Eigenvalue homomorphisms

The $E_\mu \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n}$ are eigenvectors for Cherednik-Dunkl operators $Y_1, \ldots, Y_n$:

$$Y_i E_\mu = q^{-\mu_i} t^{-(v_\mu(i)-1)+\frac{1}{2}(n-1)} E_\mu$$

Define homomorphisms $ev_\mu^t : \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}] \to \mathbb{C}$ by

$$ev_\mu^t(Y_i) = q^{-\mu_i} t^{(v_\mu(i)-1)+\frac{1}{2}(n-1)}$$

so that if $D(Y) \in \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]$ then

$$D(Y) E_\mu = ev_\mu^t(D(Y)) E_\mu.$$

<u>c-functions</u> Define $Y_{j+n} = q^{-1} Y_j$.

$$c_{ij}(Y) = \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} Y_i Y_j^{-1}}{1 - Y_i Y_j^{-1}} \quad \text{and} \quad c_w(Y) = \prod_{(i,j) \in \mathrm{Inv}(w)} c_{ij}(Y)$$

for $w \in W$, the group of n-periodic permutations.

The E_μ are constructed with <u>intertwiners</u> $\tau_\pi^\vee$ and $\tau_1^\vee, \ldots, \tau_{n-1}^\vee$:

$$E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee 1 = (\text{const})\, \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee \cdot 1$$

and $(\tau_i^\vee)^2 = c_{i,i+1}(Y)\, c_{i+1,i}(Y)$.

Normalized intertwiners

Define η_π and $\eta_{s_1}, \ldots, \eta_{s_{n-1}}$ by

$$\eta_\pi = \tau_\pi^\vee \quad \text{and} \quad \eta_{s_i} = \frac{1}{c_{i,i+1}(Y)} \tau_i^\vee .$$

Let $t_\mu : \mathbb{Z} \to \mathbb{Z}$ be the n-periodic permutation

$$t_\mu(i) = i + n\mu_i, \quad \text{for } i \in \{1, \ldots, n\}.$$

Let $v_\mu \in S_n$ be minimal length such that $v_\mu \mu$ is weakly increasing.

Let $u_\mu = t_\mu v_\mu^{-1}$ (so that $t_\mu = u_\mu v_\mu$). Define

$$N_\mu = ev_0^t(c_{u_\mu}(Y)) \quad \text{and} \quad \hat{E}_\mu = \frac{1}{N_\mu} E_\mu$$

Then

$$\eta_{s_i} \hat{E}_\mu = \begin{cases} \hat{E}_{s_i \mu}, & \text{if } \mu_i \neq \mu_{i+1} \\ 0, & \text{if } \mu_i = \mu_{i+1} \end{cases}$$

and if $w \in W$ then

$$\eta_w \hat{E}_\mu \text{ equals } \hat{E}_{w\mu} \text{ or } 0.$$

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Universal formalism

As operators on $\mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$

$x_i = T_{i-1}\cdots T_1 \tau_\pi^\vee T_{n-1}^{-1}\cdots T_i^{-1}$ so $x_1^{a_1}\cdots x_n^{a_n}$ is a product of T_i operators

$T_i = \tau_i^\vee + (c_{i+1,i}(Y) - t^{\frac{1}{2}})$ so T_i is a combination of $\tau_i^\vee$ and $c_{ij}(Y)$

$\tau_w^\vee c_{ij}(Y) = c_{w(i),w(j)}(Y)\tau_w^\vee$. so $c_{ij}(Y)$ can always be moved right of τ_w.

Thus, if $f\in\mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$ one can calculate $D_w(Y)$ such that

(*) $\quad f = \sum_{w\in W} m_w D_w(Y),\quad$ as operators on $\mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$.

Then

$$fE_\mu = \sum_{w\in W} ev_\mu^t(D_w(Y))\, ev_0^t\left(\frac{c_{u_\mu}(Y)}{c_{u_{w\mu}}(Y)}\right) E_{w\mu}$$

So (*) is a <u>universal formula</u> for multiplying f by <u>any</u> E_μ.

Monk rule: operator form

Theorem Let $j \in \{1, \ldots, n\}$. As operators on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$,

$$x_j = \sum_{\substack{C \subseteq \{1, \ldots, n\} \\ j \in C}} \tau^\vee_{C,j} \, F_{C,j}(Y) f_C(Y),$$

where, if $C = \{a_1, \ldots, a_m\}$ with

$$a_1 < \cdots < a_m \quad \text{and} \quad j = a_p$$

then

$$f_C(Y) = \frac{t^{-(m-1)/2}}{1 - q Y_{a_1} Y_{a_m}^{-1}} \left(\prod_{i=1}^{m-1} \frac{1-t}{1 - Y_{a_i} Y_{a_{i+1}}^{-1}} \right)$$

$$F_{C,j}(Y) = \begin{cases} 1 - q Y_{a_1} Y_{a_m}^{-1}, & \text{if } p = 1, \\ Y_{a_1} Y_{a_p}^{-1} - Y_{a_1} Y_{a_{p-1}}^{-1}, & \text{if } p \neq 1, \end{cases}$$

$$\tau^\vee_{C,j} = t^\vee_{b_r} t^\vee_{b_{r-1}} \cdots t^\vee_{b_1} t^\vee_{\pi} t^\vee_{b_{n-m}} \cdots t^\vee_{b_{r+1}}$$

where the complement of C,

$$C^c = \{b_1, \ldots, b_{n-m}\} \text{ with } b_1 < \cdots < b_r < j < b_{r+1} < \cdots < b_{n-m}.$$

Monk formula

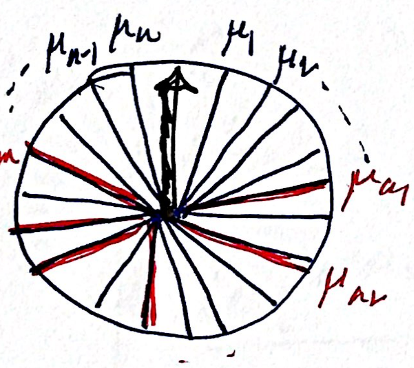
Theorem Let $j \in \{1, \ldots, n\}$ and $\mu \in \mathbb{Z}^n$. Then

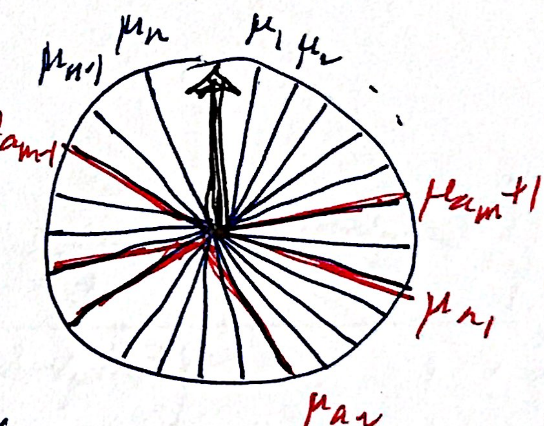
$$x_j E_\mu = \sum_{\substack{C \subseteq \{1,\ldots,n\} \\ j \in C}} F_\mu(C,j)\, wt_\mu(C)\, E_{rot_C(\mu)}$$

where, if $C = \{a_1, \ldots, a_m\}$ with

$$a_1 < \cdots < a_m \quad \text{and} \quad j = a_p$$

then

$\mu =$ 

$rot_C(\mu) =$ 

so that, in $rot_C(\mu)$, the parts of μ indexed by the elements of C have been rotated and 1 has been added to μ_{a_m},

$$F_\mu(C,j) = \begin{cases} 1 - q^{\mu_{a_m} - \mu_{a_1} + 1} t^{v_\mu(a_m) - v_\mu(a_1)}, & \text{if } p = 1, \\ q^{\mu_{a_p} - \mu_{a_1}} t^{v_\mu(a_p) - v_\mu(a_1)} - q^{\mu_{a_{p-1}} - \mu_{a_1}} t^{v_\mu(a_{p-1}) - v_\mu(a_1)}, & \text{if } p \neq 1, \end{cases}$$

and

$$wt_\mu(C) = t^{\frac{1}{2}((n-1) - \#\{\mu_i > \mu_{a_m}\})} \left(\prod_{i=1}^{n} wt_\mu(C,k) \right)$$

where the weight for μ_{a_i} is

$$\mathrm{wt}_\mu(C, a_i) = \begin{cases} \dfrac{1-t}{1-q^{\mu_{a_{i+1}}-\mu_{a_i}} t^{v_\mu(a_{i+1})-v_\mu(a_i)}}, & \text{if } i \neq m, \\[2ex] \dfrac{t^{-(m-1)/2}}{1-q^{\mu_{a_m}-\mu_{a_1}+1} t^{v_\mu(a_m)-v_\mu(a_1)}}, & \text{if } i = m, \end{cases}$$

and the "passing weight" for μ_{a_i} past μ_k is

$$\mathrm{wt}_\mu(C, k) = \begin{cases} 0, & \text{if } \mu_{a_i} = \mu_k \\ t^{\frac{1}{2}}, & \text{if } \mu_{a_i} > \mu_k \\ \dfrac{t^{\frac{1}{2}}\left(1-q^{\mu_k-\mu_{a_i}} t^{v_\mu(k)-v_\mu(a_i)+1}\right)\left(1-q^{\mu_k-\mu_{a_i}} t^{v_\mu(k)-v_\mu(a_i)-1}\right)}{\left(1-q^{\mu_k-\mu_{a_i}} t^{v_\mu(k)-v_\mu(a_i)}\right)^2}, & \text{if } \mu_{a_i} < \mu_k \end{cases}$$

if $i < m$,

and a similar weight when $i = m$ except with μ replaced by $\mathrm{rot}_2(\mu)$.