06.04.2022
Lecture 7 1

Boson-Fermion correspondences

Theorem 1 As $\mathbb{C}[X]^{S_n}$-modules

$$p_0\mathbb{C}[X] = \mathbb{C}[X]^{S_n} \xrightarrow{\sim} \mathbb{C}[X]^{\det} = e_0\mathbb{C}[X] = a_\rho\, \mathbb{C}[X]^{S_n}$$

$$f \longmapsto a_\rho f$$

$$p_0 x^\lambda \overset{\circ}{=} m_\lambda$$

$$s_\lambda \longleftrightarrow a_{\lambda+\rho} = e_0 x^{\lambda+\rho}$$

Theorem 2 As $\mathbb{C}[X]^{S_n}$-modules

$$\mathbb{C}[X]^{S_n} = \mathcal{Y}_0\mathbb{C}[X] = \mathbb{C}[X]^{Bos} \xrightarrow{\sim} \mathbb{C}[X]^{Fer} = \mathcal{E}_0\, \mathbb{C}[X] = A_\rho\, \mathbb{C}[X]^{S_n}$$

$$f \longmapsto A_\rho f$$

$$\mathcal{Y}_0 E_\lambda = P_\lambda(q,t)$$

$$P_\lambda(q,qt) \longleftrightarrow A_{\lambda+\rho} = \mathcal{E}_0 E_{\lambda+\rho}$$

Theorem 3 As Heis-modules

$$\mathbb{C}[x_1,x_2,\dots]^{S_\infty} = B \xrightarrow{\sim} F = \Lambda^{\infty/2}(\mathbb{C}^{\mathbb{Z}})$$

$$p_\mu \longmapsto \mathcal{X} p_\mu$$

$$s_\lambda \longleftrightarrow |\lambda+\rho\rangle$$

Symmetrizers for S_n

The symmetric group S_n acts on polynomials

$$\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

by
$$(s_i f)(x_1, \ldots, x_n) = f(x_1, \ldots x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)$$

for $i \in \{1, \ldots, n-1\}$. Define

$$\mathbb{C}[X]^{S_n} = \{f \in \mathbb{C}[X] \mid s_i f = f \text{ for } i \in \{1, \ldots, n-1\}\}$$
$$\mathbb{C}[X]^{\det} = \{f \in \mathbb{C}[X] \mid s_i f = -f \text{ for } i \in \{1, \ldots, n-1\}\}$$

For $w \in S_n$ (a bijection $w: \{1, \ldots, n\} \to \{1, \ldots, n\}$)

$$\mathrm{Inv}(w) = \left\{ (i,j) \;\middle|\; \begin{array}{l} i,j \in \{1, \ldots, n\}, \\ i < j \text{ and } w(i) > w(j) \end{array} \right\}$$

$$\ell(w) = \# \mathrm{Inv}(w)$$

The element $w_0 \in S_n$ is given by $w_0(i) = n-i$,

$$\mathrm{Inv}(w_0) = \{(i,j) \mid i < j\} \text{ and } \ell(w_0) = \tfrac{1}{2} n(n-1).$$

Define symmetrizers

$$p_0 = \sum_{w \in S_n} w \quad \text{and} \quad e_0 = \sum_{w \in S_n} (-1)^{\ell(w_0) - \ell(w)} w$$

so that $s_i p_0 = p_0$ and $s_i e_0 = -e_0$. Then

$$\mathbb{C}[X]^{S_n} = p_0 \mathbb{C}[X] \quad \text{and} \quad \mathbb{C}[X]^{\det} = e_0 \mathbb{C}[X].$$

Symmetrizers for the Hecke algebra

The finite Hecke algebra H_0 acts on polynomials by

$$T_i = -t^{-\frac{1}{2}} + t^{\frac{1}{2}}(1+s_i)\frac{1-t x_i x_{i+1}^{-1}}{1-x_i x_{i+1}^{-1}} \quad \text{for } i \in \{1,\dots,n-1\}.$$

Define

$$\mathbb{C}[X]^{Bos} = \{f \in \mathbb{C}[X] \mid T_i f = t^{\frac{1}{2}} f \text{ for } i \in \{1,\dots,n-1\}\}$$

$$\mathbb{C}[X]^{Fer} = \{f \in \mathbb{C}[X] \mid T_i f = -t^{-\frac{1}{2}} f \text{ for } i \in \{1,\dots,n-1\}\}.$$

A reduced word for $w \in S_n$ is an expression

$$w = s_{i_1} \cdots s_{i_\ell} \quad \text{with } \ell = \ell(w)$$

Define

$$T_w = T_{i_1} \cdots T_{i_\ell} \quad \text{if } w = s_{i_1} \cdots s_{i_\ell} \text{ is a reduced word.}$$

Define symmetrizers

$$\mathbb{1}_0 = \sum_{w \in S_n} (-1)^{\ell(w_0)-\ell(w)} T_w \quad \text{and} \quad \varepsilon_0 = \sum_{w \in S_n} (-t^{-\frac{1}{2}})^{\ell(w_0)-\ell(w)} T_w$$

so that $T_i \mathbb{1}_0 = t^{\frac{1}{2}} \mathbb{1}_0$ and $T_i \varepsilon_0 = -t^{-\frac{1}{2}} \varepsilon_0$.

Then

$$\mathbb{C}[X]^{Bos} = \mathbb{1}_0 \mathbb{C}[X] \quad \text{and} \quad \mathbb{C}[X]^{Fer} = \varepsilon_0 \mathbb{C}[X]$$

Weyl denominators

Define $\rho = (n-1, n-2, \dots, 2, 1, 0)$.

The Vandermonde determinant is

$$a_\rho = \prod_{i<j} x_j - x_i = \det \begin{pmatrix} x_1^{n-1} & x_2^{n-2} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \vdots & & & \\ x_1^{0} & x_2^{0} & \cdots & x_n^{0} \end{pmatrix}$$

$$= \det(x_j^{n-i}) = \varepsilon_0 x_1^{n-1} x_2^{n-2} \cdots x_{n-2}^{2} x_{n-1}^{1} x_n^{0} = \varepsilon_0 X^\rho.$$

The elliptic Weyl denominator is

$$A_\rho = \varepsilon_0 E_\rho = \varepsilon_0 X^\rho = \prod_{i<j} x_j - t x_i$$

where $E_\rho = E_{(n-1, n-2, \dots, 2, 1, 0)}$ is the electronic Macdonald polynomial.

Theorem

$$\mathbb{C}[X]^{S_n} = \mathbb{C}[X]^{S_n} \qquad \mathbb{C}[X]^{\det} = a_\rho \mathbb{C}[X]^{S_n}$$

$$\mathbb{C}[X]^{Bos} = \mathbb{C}[X]^{S_n} \qquad \mathbb{C}[X]^{Fer} = A_\rho \mathbb{C}[X]^{S_n}$$

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The Cherednik-Macdonald Inner product lecture 7.

Involution: Define $\bar{\ }: \mathbb{C}[X] \to \mathbb{C}[X]$ by

$$\bar{f}(x_1, \dots, x_n; q, t) = f(x_1^{-1}, \dots, x_n^{-1}; q^{-1}, t^{-1})$$

Pochammer symbols: Define

$$(a; z)_k = (1-a)(1-za)(1-z^2 a) \cdots (1-z^{k-1} a)$$

$$(a; z)_\infty = (1-a)(1-za)(1-z^2 a) \cdots$$

Kernels: Define

$$\nabla_{q,t} = \prod_{i \neq j} \frac{(x_i x_j^{-1}; q)_\infty}{(t x_i x_j^{-1}; q)_\infty} \quad \text{and} \quad \Delta_{q,t} = \nabla_{q,t} \prod_{i<j} \frac{x_j - t x_i}{x_j - x_i}$$

Inner product: Define $\langle \, , \, \rangle_{q,t}: \mathbb{C}[X] \times \mathbb{C}[X] \to \mathbb{C}$ by

$$\langle f_1, f_2 \rangle_{q,t} = ct(f_1 \bar{f_2} \Delta_{q,t})$$

where $ct(f)$ is the coefficient of $x^0 = x_1^0 \cdots x_n^0$ in f.

Characterization of Macdonald polynomials

$$(\mathbb{Z}^n)^+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}$$

is an index set for the S_n-orbits on $\mathbb{Z}^n$

$\mathbb{C}[X]$ has basis $\{x^\gamma \mid \gamma \in \mathbb{Z}^n\}$

$\mathbb{C}[X]^{S_n}$ has basis $\{m_\lambda \mid \lambda \in (\mathbb{Z}^n)^+\}$.

(A) Define an order on $(\mathbb{Z}^n)^+$ by requiring

$(\lambda_1, \ldots, \lambda_n) < (\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots, \lambda_n)$ for $i < j$.

<u>Theorem</u> P_λ is the unique element of $\mathbb{C}[X]^{S_n}$ such that

(a) $P_\lambda = m_\lambda + \sum_{\gamma < \lambda} a_\lambda^\gamma m_\gamma$

(b) If $\gamma \in (\mathbb{Z}^n)^+$ and $\gamma < \lambda$ then $(P_\lambda, m_\gamma)_{q,t} = 0$.

(B) Define an order on $\mathbb{Z}^n$ by requiring

$(\mu_1, \ldots, \mu_n) < (\mu_1, \ldots, \mu_{i+1}, \mu_i, \ldots, \mu_n)$ if $\mu_i > \mu_{i+1}$

$(\mu_1, \ldots, \mu_n) < (\mu_n + 1, \mu_1, \ldots, \mu_{n-1})$.

Theorem E_μ is the unique element of $\mathbb{C}[X]$ such that

(a) $E_\mu = x^\mu + \sum_{\nu < \mu} b_\mu^\nu x^\nu$

(b) If $\nu \in \mathbb{Z}^n$ and $\nu < \mu$ then $(E_\mu, x^\nu)_{q,t} = 0$.

The Weyl character formula

$$W_0(t) = \sum_{w \in S_n} t^{l(w)}$$

Theorem (Raising the level)

Assume $f_1, f_2 \in \mathbb{C}[X]^{S_n}$. Then

$$(f_1, f_2)_{q,qt} = t^{l(w_0)} \frac{W_0(qt)}{W_0(t)} (A_\rho f_1, A f_2)_{q,t}$$

Theorem (Weyl character formula)

Let $\lambda \in (\mathbb{Z}^n)^+$. Then

$$P_\lambda(q, qt) = \frac{A_{\lambda+\rho}(q,t)}{A_\rho(q,t)}$$

The Heisenberg Boson-Fermion Correspondlecture 7

The Heisenberg algebra is

$$\mathfrak{H} = \mathrm{span}\{s_m \mid m \in \mathbb{Z}\} + \mathbb{C}K.$$

with relations

$$[s_m, s_n] = m\,\delta_{m,-n} K, \quad \text{where } [a,b] = ab - ba.$$

The Heisenberg algebra acts on polynomials

$$B = \mathbb{C}[p_1, p_2, \ldots; q, q^{-1}] \quad \text{by}$$

$$s_m = \frac{\partial}{\partial p_m} \quad \text{and} \quad s_{-m} = m p_m \quad \text{for } m \in \{1, 2, \ldots\}$$

$$\text{and} \quad s_0 = q\frac{\partial}{\partial q} \quad \text{and} \quad K = 1.$$

The Fock space is

$$F = \mathrm{span}\left\{ i_1 \wedge i_2 \wedge \cdots \;\middle|\; \begin{array}{l} i_k \in \mathbb{Z},\ i_1 > i_2 > \cdots \text{ and} \\ \text{there exists } N \in \mathbb{Z}_{>0} \text{ such} \\ \text{that } i_n = i_{n-1} - 1 \text{ for } n \in \mathbb{Z}_{>N} \end{array} \right\}$$

For $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathbb{Z}^\infty$ with $\lambda_1 \geq \lambda_2 \geq \cdots$ and there exists $N \in \mathbb{Z}_{>0}$ such that $\lambda_n = \lambda_{n-1}$ for $n \in \mathbb{Z}_{>N}$ define

$$|\lambda + \rho_m\rangle = i_1 \wedge i_2 \wedge \cdots \quad \text{where } i_k = \lambda_k + m - k$$

$$\text{and} \quad \rho_m = (m-1, m-2, m-3, \ldots).$$

Define operators ψ_j and ψ_j^* on F, for $j \in \mathbb{Z}$, by

$$\psi_j(|\lambda+\rho_m\rangle) = \begin{cases} 0, & \text{if there exists } s \in \mathbb{Z}_{>0} \text{ with } i_s = j, \\ (-1)^s i_1 \wedge \cdots \wedge i_s \wedge j \wedge i_{s+1} \wedge \cdots & \text{if } i_s > j > i_{s+1}. \end{cases}$$

and

$$\psi_j^*(|\lambda+\rho_m\rangle) = \begin{cases} 0, & \text{if } j \neq i_s \text{ for } s \in \mathbb{Z}_{>0}, \\ (-1)^{s-1} i_1 \wedge \cdots \wedge i_{s-1} \wedge i_{s+1} \wedge \cdots & \text{if } j = i_s. \end{cases}$$

The Heisenberg algebra acts on F by

$$s_m = \sum_{j \in \mathbb{Z}} \psi_j \psi_{j+m}^*, \quad \text{for } m \neq 0,$$

$$s_0 = \sum_{j \in \mathbb{Z}_{>0}} \psi_j \psi_j^* - \sum_{j \in \mathbb{Z}_{\leq 0}} \psi_j^* \psi_j$$

<u>Theorem</u> As Heis-modules

$$B \xrightarrow{\sim} F$$

$$q^m \longmapsto |\rho_m\rangle$$

$$q^m p_\mu \longmapsto \sum_\lambda \chi_{S_n}^\lambda(\mu)\, |\lambda+\rho_m\rangle$$

$$q^m s_\lambda \longleftarrow\!\!\shortmid\ |\lambda+\rho_m\rangle$$