23.02.2022
Macdonald ①

Lecture 1: Permutations

Let $n \in \mathbb{Z}_{>0}$, ie. $n=5$.

A <u>permutation</u> is a bijection

$$v : \{1, \ldots, n\} \to \{1, \ldots, n\}.$$

The <u>symmetric group</u> S_n is the group of permutations with operation composition.

Let $v \in S_n$. The set of <u>inversions of v</u> is

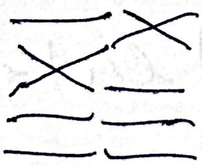
$$Inv(v) = \left\{ (i,j) \;\middle|\; \begin{array}{l} i,j \in \{1,\ldots,n\} \\ i<j \text{ and } v(i) > v(j) \end{array} \right\}$$

$\ell(v) = \# Inv(v)$. Write $(i,j) \in \varepsilon_i^v - \varepsilon_j^v$

The <u>simple reflections</u> $s_1, \ldots, s_{n-1}$ are given by

$s_i(i) = i+1$

$s_i(i+1) = i$ and $s_i(j) = j$ for $j \notin \{i, i+1\}$.

<u>Example</u> $s_2 s_1 =$  is $(s_2s_1)(1) = 3$

$(s_2s_1)(2) = 1$

$(s_2s_1)(3) = 2$ with

$(s_2s_1)(4) = 4$

$(s_2s_1)(5) = 5$

$$Inv(s_2s_1) = \{\varepsilon_1^v - \varepsilon_2^v, \varepsilon_1^v - \varepsilon_3^v\} = \{(1,2), (1,3)\}$$

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A n-periodic permutation is a bijection

$w: \mathbb{Z} \to \mathbb{Z}$ such that $w(i+n) = w(i)+n$.

The affine Weyl group W is the group of n-periodic permutations with operation composition.

Let $w \in W$. Define

$$\mathrm{Inv}(w) = \left\{ (i,k) \;\middle|\; \begin{array}{l} i \in \{1,\dots,n\},\ k \in \mathbb{Z} \\ i < k \text{ and } w(i) > w(k) \end{array} \right\}$$

$\ell(w) = \#\mathrm{Inv}(w)$. Write $(i, j+\ell n) = \varepsilon_i^\vee - \varepsilon_j^\vee + \ell K$ for $i,j \in \{1,\dots,n\}$ and $\ell \in \mathbb{Z}$

The simple reflections are $s_0, s_1, \dots, s_{n-1}$ given by

$$\begin{array}{l} s_i(i) = i+1 \\ s_i(i+1) = i \end{array} \quad \text{and } s_i(j) = j \text{ for } j \in \{1,\dots,n\} \text{ with } j \notin \{i, i+1\}.$$

Let $\pi: \mathbb{Z} \to \mathbb{Z}$ be given by

$$\pi(i) = i+1.$$

For $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ define $t_\mu: \mathbb{Z} \to \mathbb{Z}$ by

$$t_\mu(i) = i + n\mu_i$$

The finite Weylgroup $W_0 = S_n$ is the subgroup of W generated by $s_1, \dots, s_{n-1}$.

Let $\mu \in \mathbb{Z}^n_{\geq 0}$ with $\mu = (\mu_1, \mu_2, \dots, \mu_n)$

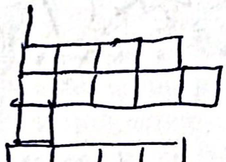
A box in μ is (r,c) with

$$r \in \{1, \dots, n\} \quad \text{and} \quad c \in \{1, \dots, \mu_r\}.$$

i.e.

$$\{\text{boxes in } \mu\} = \{(r,c) \mid r \in \{1, \dots, n\},\ c \in \{1, \dots, \mu_r\}\}.$$

Example

$\mu = (0,4,5,1,4) =$ 

Let $v_\mu \in S_n$ be minimal length such that

$v_\mu \mu$ is weakly increasing

Define $u_\mu \in W$ by

$$u_\mu = t_\mu v_\mu^{-1}$$

Theorem (Guo-Ram) Let $\mu \in \mathbb{Z}^n_{\geq 0}$. Let $(r,c) \in \mu$

$$u_\mu(r,c) = \#\{r' < r \mid (r',c) \notin \mu\} \cup \#\{r' > r \mid (r', c-1) \notin \mu\}.$$

Then

$$u_\mu = \prod_{(r,c) \in \mu} (s_{u_\mu(r,c)} \cdots s_2 s_1 \pi),$$

$$Inv(u_\mu) = \bigcup_{(r,c) \in \mu} \bigcup_{i=1}^{u_\mu(r,c)} \{\varepsilon^\vee_{v_\mu(r)} - \varepsilon^\vee_i + (\mu_r - c + 1)K\}$$

$$\ell(u_\mu) = \sum_{(r,c) \in \mu} u_\mu(r,c)$$

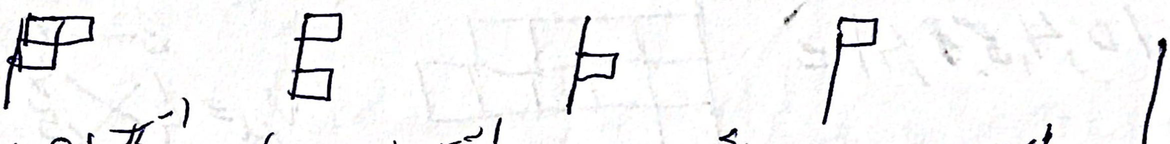
W-action on $\mathbb{Z}^n$ Define

$$\pi(\mu_1,\dots,\mu_n) = (\mu_n+1, \mu_1, \mu_2, \dots, \mu_{n-1})$$

$$s_i(\mu_1,\dots,\mu_n) = (\mu_1,\dots,\mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2},\dots,\mu_n)$$

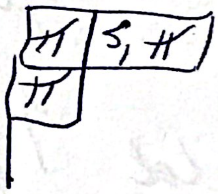
for $i \in \{1,\dots,n-1\}$.

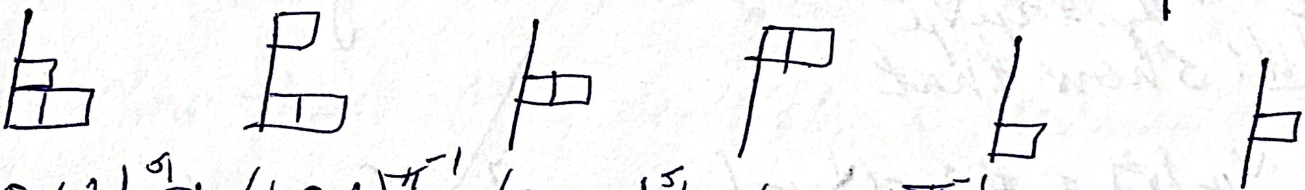
Examples



$$(2,1,0) \xrightarrow{\pi^{-1}} (1,0,1) \xrightarrow{\pi^{-1}} (0,1,0) \xrightarrow{s_1} (1,0,0) \xrightarrow{\pi^{-1}} (0,0,0)$$

So $(2,1,0) = \pi\pi s_1\pi(0,0,0)$ $\quad u_{(2,1,0)} = \pi\pi s_1\pi =$

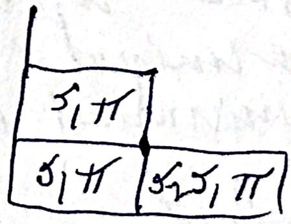




$$(0,1,2) \xrightarrow{s_1} (1,0,2) \xrightarrow{\pi^{-1}} (0,2,0) \xrightarrow{s_1} (2,0,0) \xrightarrow{\pi^{-1}} (0,0,1) \xrightarrow{s_2} (0,1,0)$$

So $(0,1,2) = s_1\pi s_1\pi s_2 s_1\pi(0,0,0)$

$u_{(0,1,2)} = s_1\pi\, s_1\pi\, s_2s_1\pi =$



HW Show that

$$W_0 = \mathrm{Stab}(0,0,\dots,0)$$
$$= \{w \in W \mid w(0,0,\dots,0) = (0,0,\dots,0)\} = S_n$$

HW Show that
$$\{s_1,\dots,s_{n-1}\}=\{v\in S_n \mid \ell(v)=1\}.$$
HW: Show that if $w\in W$ then $\ell(w)$ is finite

HW: Show that $\ell(\pi)=0$ and
$$\ell(s_i)=1 \text{ for } i\in\{0,1,\dots,n-1\}.$$
HW: Show that
$$\{\pi^k \mid k\in\mathbb{Z}\}=\{w\in W \mid \ell(w)=0\}.$$
HW: Show that the subgroup W^{ad} of W generated by $\{s_0,s_1,\dots,s_{n-1}\}$ is
$$W^{ad}=\{w\in W \mid w(1)+\dots+w(n)=\tfrac{1}{2}n(n-1)\}$$
HW: Show that it is sensible to define
$$s_i=s_{i+n} \text{ for } i\in\mathbb{Z}$$
and then
$$s_i^2=1, \qquad s_is_{i+1}s_i=s_{i+1}s_is_{i+1}$$
$$\pi s_i\pi^{-1}=s_{i+1}, \text{ and } s_is_j=s_js_i \text{ if } j\notin\{i-1,i+1\}.$$
HW: Show that if $\mu\in\mathbb{Z}^n$ and $v\in S_n$ then
$$vt_\mu=t_{v\mu}\,\mu$$

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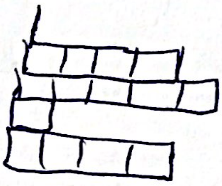
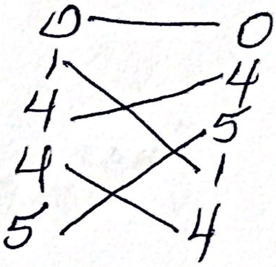
HW: Show that $v_{(0,4,5,1,4)} = s_4 s_2 s_3$

$\mu =$ 

$v_{(0,4,5,1,4)} =$ 

HW: Show that

$$v_\mu(r) = \#\{r' > r \mid \mu_{r'} < \mu_r\} + \#\{r' < r \mid \mu_{r'} \le \mu_r\}$$

HW: Show that $u_\mu W_0 = t_\mu W_0$ and u_μ is the unique minimal length element in the coset $t_\mu W_0$.

HW: Show that

$$t_\mu = u_\mu v_\mu \text{ and } \ell(t_\mu) = \ell(u_\mu) + \ell(v_\mu)$$

HW: Show that

$$Inv(v_\mu) = \#\{(i,j) \mid i<j \text{ and } \mu_i > \mu_j\}$$

HW: Show that if $\mu \in \mathbb{Z}^n$ and $v \in S_n$ then $v t_\mu = t_{v\mu} v$.

HW: Is it true that $\pi t_\mu = t_{\pi\mu} \pi$?

HW: Show that
$$W = \{t_\mu v \mid \mu \in \mathbb{Z}^n, v \in S_n\}$$

HW: Show that
$$W = \{\pi^k u \mid k \in \mathbb{Z}, u \in W^{ad}\}$$

HW: Show that $\pi^n = t_{(1,1,\ldots,1)}$ and
$$Z(W) = \{\pi^{ln} \mid l \in \mathbb{Z}\}.$$

HW: Let $\varepsilon_i = (0,\ldots,0,1,0,\ldots,0)$ with 1 in the i^{th} spot. Show that $t_{(1,0,\ldots,0)} = \pi s_{n-1}\cdots s_1$
$$t_{\varepsilon_i} = s_{i-1}\cdots s_2 s_1 \pi s_{n-1}\cdots s_i$$

<u>Example</u> Let $\mu = (0,4,5,1,4)$

Then
$$u_\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 9 & 22 & 25 & 28 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4+n & 2+4n & 5+4n & 3+5n \end{pmatrix}$$

$$v_\mu = s_4 s_2 s_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$u_\mu = (s_1\pi)^6 (s_2 s_1 \pi)^7 (s_3 s_2 s_1 \pi)$$

$$l(u_\mu) = 23, \quad l(v_\mu) = 3, \quad l(t_\mu) = 26$$