1.18 Series

Let X be a topological group with operation addition and let $(a_1, a_2, a_3, \ldots)$ be a sequence in X.

- The *series* $\sum_{n=1}^{\infty} a_n$ is the sequence $(s_1, s_2, s_3, \ldots)$,

$$\text{where } s_k = a_1 + a + 2 + \cdots + a_k. \qquad \text{Write} \quad \sum_{n=1}^{\infty} a_n = \ell \quad \text{if} \quad \lim_{n\to\infty} s_n = \ell.$$

- The series $\sum_{n=1}^{\infty} a_n$ *converges in* X if the sequence $(s_1, s_2, s_3, \ldots)$ converges in X.
- The series $\sum_{n=1}^{\infty} a_n$ *diverges in* X if the sequence $(s_1, s_2, s_3, \ldots)$ diverges in X.

1.18.1 Root and ratio tests for convergence

Theorem 1.10. *(Root and ratio tests) Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$.*

(a) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ *exists and* $a < 1$ *then* $\sum_{n=1}^{\infty} |a_n|$ *converges in* $\mathbb{R}$.

(b) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ *exists and* $a > 1$ *then* $\sum_{n=1}^{\infty} |a_n|$ *diverges in* $\mathbb{R}$.

(c) If $\lim_{n\to\infty} |a_n|^{1/n} = a$ *exists and* $a < 1$ *then* $\sum_{n=1}^{\infty} |a_n|$ *converges in* $\mathbb{R}$.

(d) If $\lim_{n\to\infty} |a_n|^{1/n} = a$ *exists and* $a > 1$ *then* $\sum_{n=1}^{\infty} |a_n|$ *diverges in* $\mathbb{R}$.

1.18.2 Radius of convergence

Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{C}$ and let

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \text{(an element of } \mathbb{C}[[x]]\text{)}.$$

The *radius of convergence of* $\sum_{n=0}^{\infty} a_n x^n$ is

$$\mathrm{ROC}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sup\left\{ |r| \;\Big|\; r \in \mathbb{C} \text{ and } \sum_{n=0}^{\infty} a_n r^n \text{ converges} \right\}.$$

The following proposition is what ensures that the knowledge of $\mathrm{ROC}\left(\sum_{n=0}^{\infty} a_n x^n\right)$ is useful.

Proposition 1.11. *Let $(a_0, a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$ or $\mathbb{C}$. Let $r, s \in \mathbb{C}$ and*

$$\textit{assume } \sum_{n=0}^{\infty} a_n s^n \textit{ converges.} \qquad \textit{If } |r| < |s| \quad \textit{then} \quad \sum_{n=0}^{\infty} a_n |r|^n \textit{ converges.}$$