1.17 Sequences

Let Y be a set. A *sequence* $(y_1, y_2, y_3, \ldots)$ in Y is a function

$$\begin{array}{ccc} \mathbb{Z}_{>0} & \longrightarrow & Y \\ n & \longmapsto & y_n \end{array}$$

Let Y be a set with a partial order $\le$ and let $(y_1, y_2, y_3, \ldots)$ be a sequence in Y.

- The sequence $(y_1, y_2, y_3, \ldots)$ is *increasing* if $(y_1, y_2, y_3, \ldots)$ satisfies

$$\text{if } i \in \mathbb{Z}_{>0} \qquad \text{then} \quad y_i \le y_{i+1}.$$

- The sequence $(y_1, y_2, y_3, \ldots)$ is *decreasing* if $(y_1, y_2, y_3, \ldots)$ satisfies

$$\text{if } i \in \mathbb{Z}_{>0} \qquad \text{then} \quad y_i \ge y_{i+1}.$$

- The sequence $(y_1, y_2, y_3, \ldots)$ is *monotone* if it is increasing or decreasing.

Let Y be a metric space and let $(y_1, y_2, y_3, \ldots)$ be a sequence in Y.

- The sequence $(y_1, y_2, y_3, \ldots)$ is *bounded* if the set $\{y_1, y_2, y_3, \ldots\}$ is bounded.
- The sequence $(y_1, y_2, y_3, \ldots)$ is *Cauchy* if $(y_1, y_2, \ldots)$ satisfies:

$$\text{if } \varepsilon \in \mathbb{R}_{>0} \text{ then} \quad \text{there exists } N \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } m, n \in \mathbb{Z}_{\ge N} \text{ then } d(y_m, y_n) < \varepsilon.$$

- Let $\ell \in Y$. The sequence $(y_1, y_2, y_3, \ldots)$ *converges to* ℓ if

$$\lim_{n\to\infty} y_n = \ell$$

  i.e., if $(y_1, y_2, y_3, \ldots)$ satisfies

$$\text{if } \varepsilon \in \mathbb{R}_{>0} \text{ then there exists } N \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } n \in \mathbb{Z}_{\ge N} \text{ then } d(y_n, \ell) < \varepsilon.$$

- The sequence $(y_1, y_2, \ldots)$ *converges in* Y if there exists $\ell \in Y$ such that $(y_1, y_2, \ldots)$ converges to ℓ.
- The sequence $(y_1, y_2, \ldots)$ *diverges in* Y if there does not exist $\ell \in Y$ such that $(y_1, y_2, \ldots)$ converges to ℓ.

Let $(y_1, y_2, y_3, \ldots)$ be a sequence in $\mathbb{R}$.

- The *supremum* of $(y_1, y_2, y_3, \ldots)$ is the least upper bound $\sup\{y_1, y_2, y_3, \ldots\}$.
- The *infimum* of $(y_1, y_2, y_3, \ldots)$ is the greatest lower bound $\inf\{y_1, y_2, y_3, \ldots\}$.
- The *upper limit* or *limsup* of $(y_1, y_2, y_3, \ldots)$ is

$$\limsup y_n = \lim_{n\to\infty} (\sup\{y_n, y_{n+1}, \ldots\}).$$

- The *lower limit* or *liminf* of $(y_1, y_2, y_3, \ldots)$ is

$$\liminf y_n = \lim_{n\to\infty} (\inf\{y_n, y_{n+1}, \ldots\}).$$