5.10 Limits and composition of functions: proof

Theorem 5.6. (Limits and composition of functions)
Let $m, n, p \in \mathbb{Z}_{>0}$. Let Let $f\colon \mathbb{R}^n \to \mathbb{R}^p$ and $g\colon \mathbb{R}^m \to \mathbb{R}^n$ be functions and let $a \in \mathbb{R}^m$ and $\ell \in \mathbb{R}^n$.

$$\text{Assume that} \quad \lim_{x\to a} g(x) \text{ and } \lim_{x\to a} f(g(x)) \text{ exist} \qquad \text{and} \qquad \lim_{x\to a} g(x) = \ell.$$

Then

$$\lim_{y\to \ell} f(y) = \lim_{x\to a} f(g(x)).$$

Proof.

Let $L = \lim_{y\to\ell} f(y)$.
To show: $\lim_{x\to a} f(g(x)) = L$.
To show: If $e \in \mathbb{Z}_{>0}$ then there exists $d \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^m \text{ is within } 10^{-d} \text{ of } a \text{ then } f(g(x)) \text{ is within } 10^{-e} \text{ of } L.$$

Assume $e \in \mathbb{Z}_{>0}$.
To show: There exists $d \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^m \text{ is within } 10^{-d} \text{ of } a \text{ then } f(g(x)) \text{ is within } 10^{-e} \text{ of } L.$$

Since $\lim_{y\to\ell} f(y) = L$ we know that there exists $d_1 \in \mathbb{Z}_{>0}$ such that

$$\text{if } y \in \mathbb{R}^n \text{ is within } 10^{-d_1} \text{ of } \ell \text{ then } f(y) \text{ is within } 10^{-e} \text{ of } L.$$

Since $\lim_{x\to a} g(x) = \ell$ we know that there exists $d \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^m \text{ is within } 10^{-d} \text{ of } a \text{ then } g(x) \text{ is within } 10^{-d_1} \text{ of } \ell.$$

To show: If $x \in \mathbb{R}^n$ is within 10^{-d} of a then $f(g(x))$ is within 10^{-e} of L.
Assume $x \in \mathbb{R}^n$ is within 10^{-d} of a.
To show: $f(g(x)$ is within 10^{-e} of L.
Since x is within 10^{-d} of a then $g(x)$ is within 10^{-d_1} of ℓ,
and so $f(g(x))$ is within 10^{-e} of L.
So, if $x \in \mathbb{R}^n$ is within 10^{-d} of a then $f(g(x))$ is within 10^{-e} of L.
So there exists $d \in \mathbb{Z}_{>0}$ such that if $x \in \mathbb{R}^n$ is within 10^{-d} of a then $f(g(x))$ is within 10^{-e} of L.
So $\lim_{x\to a} f(g(x)) = L$.

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