5.12 Limits and order for sequences: proof

Theorem 5.8. (Limits and order for sequences) *Let* $(a_1, a_2, \ldots)$ *and* $(b_1, b_2, \ldots)$ *be sequences in* $\mathbb{R}$. *Assume that* $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *exist and*

$$\text{if } n \in \mathbb{Z}_{>0} \text{ then } a_n \le b_n.$$

Then $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.

Proof.

Let $\ell_1 = \lim_{n\to\infty} a_n$ and $\ell_2 = \lim_{n\to\infty} b_n$.
To show: If $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ satisfy the condition

$$\text{if } n \in \mathbb{Z}_{>0} \text{ then } a_n \le b_n,$$

then $\ell_1 \le \ell_2$.
Proof by contrapositive.
Assume $\ell_1 > \ell_2$ (the opposite of $\ell_1 \le \ell_2$ is $\ell_1 > \ell_2$).
To show: There exists $N \in \mathbb{Z}_{>0}$ such that $a_N > b_N$
(the opposite of 'if $n \in \mathbb{Z}_{>0}$ then $a_n \le b_n$' is 'there exists $N \in \mathbb{Z}_{>0}$ such that $a_N > b_N$').
Let $r \in \mathbb{Z}_{>0}$ be such that $10^{-r} < \ell_1 - \ell_2$.
Since $\lim_{n\to\infty} a_n = \ell_1$ then we know that there there exists $N_1 \in \mathbb{Z}_{>0}$ such that

$$\text{if } n \in \mathbb{Z}_{>0} \text{ is at least } N_1 \text{ then } a_n \text{ is within } 10^{-(r+1)} \text{ of } \ell_1.$$

Since $\lim_{n\to\infty} b_n = \ell_2$ then we know that there there exists $N_2 \in \mathbb{Z}_{>0}$ such that

$$\text{if } n \in \mathbb{Z}_{>0} \text{ is at least } N_2 \text{ then } b_n \text{ is within } 10^{-(r+1)} \text{ of } \ell_2.$$

Let $N = \max(N_1, N_2)$.
To show: $a_N > b_N$.

$$\begin{aligned} a_N &> \ell_1 - 10^{-(r+1)} = \ell_1 - \ell_2 + \ell_2 - 10^{-(r+1)} \\ &> 10^{-r} + \ell_2 - 10^{-(r+1)} > \ell_2 + 10^{-(r+1)} > b_N. \end{aligned}$$

This proves that if $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ satisfy the condition 'if $n \in \mathbb{Z}_{>0}$ then $a_n \le b_n$' then $\ell_1 \le \ell_2$.

□