5.11 Limits and order: proof

Theorem 5.7. (Limits and order) *Let $n \in \mathbb{Z}_{>0}$ and let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be functions. Let $a \in \mathbb{R}^n$. Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and*

$$\textit{if } x \in X \textit{ then } f(x) \le g(x).$$

Then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

Proof.

Let $\ell_1 = \lim_{x\to a} f(x)$ and $\ell_2 = \lim_{x\to a} g(x)$.
To show: If f and g satisfy the condition

$$\text{if } x \in X \text{ then } f(x) \le g(x),$$

then $\ell_1 \le \ell_2$.
Proof by contrapositive.
Assume $\ell_1 > \ell_2$ (the opposite of $\ell_1 \le \ell_2$ is $\ell_1 > \ell_2$).
To show: There exists $x \in \mathbb{R}^n$ such that $f(x) > g(x)$
(the opposite of 'if $x \in \mathbb{R}^n$ then $f(x) \le g(x)$' is 'there exists $x \in \mathbb{R}^n$ such that $f(x) > g(x)$.').
Let $r \in \mathbb{Z}_{>0}$ be such that $10^{-r} < \ell_1 - \ell_2$.
Since $\lim_{x\to a} f(x) = \ell_1$ then we know that there exists $d_1 \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^n \text{ is within } 10^{-d_1} \text{ of } a \text{ then } f(x) \text{ is within } 10^{-(r+1)} \text{ of } \ell_1.$$

Since $\lim_{x\to a} g(x) = \ell_2$ then we know that there exists $d_2 \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^n \text{ is within } 10^{-d_2} \text{ of } a \text{ then } f(x) \text{ is within } 10^{-(r+1)} \text{ of } \ell_2.$$

Let $d = \max(d_1, d_2)$ and let $x \in \mathbb{R}^n$ be within 10^{-d} of a (so that $x \ne a$ but x is quite close to a).
To show: $f(x) > g(x)$.

$$f(x) > \ell_1 - 10^{-(r+1)} = \ell_1 - \ell_2 + \ell_2 - 10^{-(r+1)} > 10^{-r} + \ell_2 - 10^{-(r+1)} > \ell_2 + 10^{-(r+1)} > g(x).$$

This proves that if f and g satisfy the condition 'if $x \in X$ then $f(x) \le g(x)$' then $\ell_1 \le \ell_2$.

□