5.7 Limits and addition: proof

Theorem 5.3. (Limits and addition)
Let $n \in \mathbb{Z}_{>0}$. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be functions and let $a \in \mathbb{R}^n$.

$$\text{Assume that} \quad \lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x) \qquad \text{exist.}$$

Then $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$,

Proof.

Let $l_1 = \lim_{x\to a} f(x)$ and $l_2 = \lim_{x\to a} g(x)$.
To show: $\lim_{x\to a}(f(x)+g(x)) = l_1 + l_2$.
To show: If $e \in \mathbb{Z}_{>0}$ then there exists $d \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \text{ is within } 10^{-d} \text{ of } a \text{ then } f(x)+g(x) \text{ is within } 10^{-e} \text{ of } l_1 + l_2.$$

Assume $e \in \mathbb{Z}_{>0}$.
Since $\lim_{x\to a} f(x) = l_1$ then we know that there exists $d_1 \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \text{ is within } 10^{-d_1} \text{ of } a \text{ then } f(x) \text{ is within } 10^{-(e+1)} \text{ of } l_1.$$

Since $\lim_{x\to a} g(x) = l_2$ then we know that there exists $d_2 \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \text{ is within } 10^{-d_2} \text{ of } a \text{ then } f(x) \text{ is within } 10^{-(e+1)} \text{ of } l_2.$$

Let $d = \max(d_1, d_2)$.
To show: if x is within 10^{-d} of a then $f(x)+g(x)$ is within 10^{-e} of l_1+l_2.
Assume x is within 10^{-d} of a.
To show: $f(x)+g(x)$ is within 10^{-e} of l_1+l_2.

$$\begin{aligned} |(f(x)+g(x)) - (l_1+l_2)| &= |(f(x)-l_1) + (g(x)-l_2)| \\ &\le |f(x)-l_1| + |g(x)-l_2| \\ &\le 10^{-e+1} + 10^{-(e+1)} = \frac{2}{10}10^{-e} < 10^{-e}. \end{aligned}$$

So $(f(x)+g(x)$ is within 10^{-e} of l_1+l_2.
So $\lim_{x\to a}(f(x)+g(x)) = l_1 + l_2 = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.

□