1.18.3 Absolute convergence

Proposition 1.12. *Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{K}$.*

$$\text{If} \quad \sum_{n=1}^{\infty} |a_n| \text{ converges in } \mathbb{R}_{\geq 0} \qquad \text{then} \qquad \sum_{n=1}^{\infty} a_n \text{ converges in } \mathbb{K}.$$

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{K}$.

- The series $\sum_{n=1}^{\infty} a_n$ *converges absolutely in* $\mathbb{K}$ if $\sum_{n=1}^{\infty} |a_n|$ converges in $\mathbb{R}_{\geq 0}$.
- The series $\sum_{n=1}^{\infty} a_n$ *converges conditionally in* $\mathbb{K}$ if

$$\sum_{n=1}^{\infty} |a_n| \text{ diverges in } \mathbb{R}_{\geq 0} \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n \text{ converges in } \mathbb{K}.$$

Proposition 1.13.

(a) Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{C}$ which converges absolutely in $\mathbb{C}$.

$$\text{Let } a = \sum_{n=1}^{\infty} a_n. \qquad \text{Then} \quad \text{every rearrangement of } \sum_{n=1}^{\infty} a_n \text{ converges to } a.$$

(b) Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$ which converges conditionally in $\mathbb{R}$.

$$\text{If } \ell \in \mathbb{R} \quad \text{then} \quad \text{there exists a rearrangement of } \sum_{n=1}^{\infty} a_n \text{ which converges to } \ell.$$

Proposition 1.14. *(Leibniz's theorem) If $(a_1, a_2, a_3, \ldots)$ is a decreasing sequence in $\mathbb{R}_{\geq 0}$*

$$\text{such that} \quad \lim_{n\to\infty} a_n = 0 \qquad \text{then} \qquad \sum_{n=1}^{\infty} (-1)^n a_n \text{ converges.}$$

The favorite example here is $(a_1, a_2, \ldots) = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$, which has

$$\sum_{i=1}^{\infty} (-1)^{n-1} \tfrac{1}{n} = \log 2 \qquad \text{and} \qquad \sum_{i=1}^{\infty} \left|(-1)^{n-1} \tfrac{1}{n}\right| = \sum_{i=1}^{\infty} \frac{1}{n} \quad \text{diverges.}$$