GTLA Lecture 27.08.2020

Jordan Normal Form Let $n \in \mathbb{Z}_{>0}$ and $A \in M_n(\mathbb{C})$. Then there exists $P \in GL_n(\mathbb{C})$ such that

$P^{-1}AP$ is a direct sum of Jordan blocks.

(There might be many choices for P but every possible P gives the same size of blocks and same eigenvalues.)

$$J_d(\lambda) = \underbrace{\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}}_{d \text{ columns}} \left.\vphantom{\begin{pmatrix} \lambda \\ \\ \\ 0 \end{pmatrix}}\right\} d \text{ rows}$$

Definition The matrix A is diagonalisable, or semisimple, if there exists $P \in GL_n(\mathbb{C})$ such that $P^{-1}AP$ is diagonal.

$$P^{-1}AP = \begin{pmatrix} \boxed{\lambda_1} & & & 0 \\ & \boxed{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \boxed{\lambda_n} \end{pmatrix}$$

A is semisimple if and only if all Jordan blocks of A are of size 1.

<u>Definition</u> Let $A \in M_n(\mathbb{C})$. The matrix A is <u>nilpotent</u> if there exists $k \in \mathbb{Z}_{>0}$ such that

$$A^k = 0.$$

<u>Claim</u>: A is nilpotent if and only if all Jordan blocks of A have 0 eigenvalue.

<u>Example</u>

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{matrix}} & & 0 \\ & \boxed{\begin{matrix} 0 & 1 \\ & 0 \end{matrix}} & \\ 0 & & \boxed{0} \end{pmatrix}$$

Theorem (Jordan Decomposition)
Let $n \in \mathbb{Z}_{>0}$ and $A \in M_n(\mathbb{C})$.
Then there exist $S, N \in M_n(\mathbb{C})$ such that

(a) $A = S + N$

(b) S is semisimple and N is nilpotent

(c) $SN = NS$.

This theorem says:
semisimple and nilpotent matrices control all matrices.

Proposition Let $n \in \mathbb{Z}_{>0}$ and $A \in M_n(\mathbb{C})$.
Let $P \in GL_n(\mathbb{C})$.

(a) A is nilpotent if and only if $P^{-1}AP$ is nilpotent.

(b) Let $d \in \mathbb{Z}_{>0}$ and $\lambda \in \mathbb{C}$. Let $J = J_d(\lambda)$.
Then J is nilpotent if and only if $\lambda = 0$.

Proof (a) $\Rightarrow$: Assume A is nilpotent.

To show: $P^{-1}AP$ is nilpotent.

To show: There exists $k \in \mathbb{Z}_{>0}$ such that $(P^{-1}AP)^k = 0$.

Let $k \in \mathbb{Z}_{>0}$ be such that $A^k = 0$.

To show: $(P^{-1}AP)^k = 0$.

$$(P^{-1}AP)^k = \underbrace{P^{-1}A\cancel{P} \cdot \cancel{P^{-1}}A\cancel{P} \cdots \cancel{P^{-1}}AP}_{k \text{ times.}}$$

$$= P^{-1}\underbrace{AA\cdots A}_{k \text{ times}}P = P^{-1}A^kP$$

$$= P^{-1} \cdot 0 \cdot P = 0.$$

(a) $\Leftarrow$: Assume $P^{-1}AP$ is nilpotent.

To show: A is nilpotent.

To show: There exists $k \in \mathbb{Z}_{>0}$ such that $A^k = 0$.

Let $k \in \mathbb{Z}_{>0}$ be such that $(P^{-1}AP)^k = 0$

To show: $A^k = 0$.

$$A^k = \underbrace{PP^{-1}APP^{-1}APP^{-1}A \cdots PP^{-1}A}_{k \text{ times.}}$$

$$= (PP^{-1}A \cdots PP^{-1}A)\cdot PP^{-1}$$

$$= P(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)P^{-1}$$

$$= P\left((P^{-1}AP)^k\right)P^{-1} = P\cdot 0\cdot P^{-1} = 0.$$

So A is nilpotent.

(b) $\Rightarrow$ Let $J = J_d(\lambda)$.

To show: If J is nilpotent then $\lambda = 0$.

To show: If $\lambda \neq 0$ then J is not nilpotent.

Assume $\lambda \neq 0$.

If $k \in \mathbb{Z}_{>0}$ then $1 = \lambda^{-k}\lambda^{k} \neq 0\cdot\lambda^{-k} = 0$ gives $\lambda^k \neq 0$. Using that $\mathbb{F}$ is a field

To show: J is not nilpotent.

To show: If $k \in \mathbb{Z}_{>0}$ then $J^k \neq 0$.

Assume $k \in \mathbb{Z}_{>0}$.

$$J^k = J_d(\lambda)^k = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}^k$$

$$\left(\begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}^2 = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & & \\ & & \\ & & \end{pmatrix}\right)$$

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}^k = \begin{pmatrix} \lambda^k & & \\ & & \\ & & \end{pmatrix}.$$

The $(1,1)$ entry of $J_d(\lambda)^k$ is λ^k.

Since $\lambda^k \neq 0$ then $J^k = J_d(\lambda)^k \neq 0$.

So J is not nilpotent.

(b) $\Leftarrow$: To show: If $\lambda = 0$ then

$J = J_d(\lambda)$ is nilpotent.

Assume $\lambda = 0$.

To show: There exists $k \in \mathbb{Z}_{>0}$ such that $J^k = 0$.

Let $k = d+1$.

To show: $J^k = 0$.

$$J^k = J^{d+1} = J_d(\lambda)^{d+1} = J_d(0)^{d+1}$$

$$= \underbrace{\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}}_{d \text{ times}}$$

$$= \begin{pmatrix} 0 & 0 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 & 0 \\ & \ddots & \ddots \\ & & 0 \end{pmatrix}}_{d-1 \text{ times}}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ & \ddots & & \ddots & 1 \\ & & \ddots & & 0 \\ & & & \ddots & 0 \\ & & & & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}}_{d-2 \text{ times.}}$$

After $d-1$ multiplications we get

$$\begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ 0 & & & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} = 0.$$

So $J^k = 0$.

So J is nilpotent. //

Summary:

- A is nilpotent if and only if all Jordan blocks have eigenvalue 0.
- A is semisimple if and only if a Jordan blocks have size 1.

Theorem (Jordan decomposition)

If $A \in M_n(\mathbb{C})$ then there exist $S, N \in M_n(\mathbb{C})$ such that

(a) $A = S + N$
(b) S is semisimple and
N is nilpotent, and
(c) $SN = NS$.

Sketch of proof Using Jordan Normal form
Let $P \in GL_n(\mathbb{C})$
such that
$P^{-1}AP$ is a direct sum of
Jordan blocks.
If $J_d(\lambda)$ is one of these blocks

$$J_d(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

$$= C_\lambda + J_d(0)$$

Let C be the direct sum of C_λ
one for each Jordan block
of A.
Let X be the direct sum of $J_d(0)$
one for each Jordan block
of A.

Let $S = PCP^{-1}$ and $N = PXP^{-1}$.

Then

$$A = P(P^{-1}AP)P^{-1}.$$

$$= P(C + X)P^{-1}$$

$$= PCP^{-1} + PXP^{-1} = S + N.$$

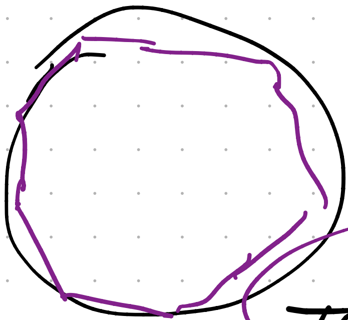
$$C = C_\lambda^{(1)} \oplus \cdots \oplus C_\lambda^{(k)}$$

$$X = J_{d_1}(0) \cdots \oplus J_{d_k}(0)$$

Then $P^{-1}AP$

$$= C + X = (C_\lambda^{(1)} + J_{d_1}(0)) \oplus \cdots \oplus (C_\lambda^{(k)} \oplus J_{d_k}(0))$$

$$= J_{d_1}(\lambda) \oplus \cdots \oplus J_{d_k}(\lambda).$$



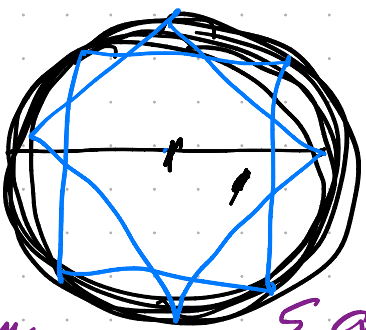
$\frac{a_5}{b_5}$ $\frac{a_6}{b_6}$ $\frac{a_7}{b_7}$

$\sqrt{1^2+1^2} = \sqrt{2}$

$\pi \notin \mathbb{Q}$

$\pi \notin \overline{\mathbb{Q}}$

alg. closure.



$$\frac{\text{circumference}}{\text{diameter}} = \pi$$

$\frac{\text{circum}}{\text{diameter}} =$ 5 gon $\frac{a_5}{b_5}$ 7 gon $\frac{a_7}{b_7}$ 10 gon $\frac{a_{10}}{b_{10}}$